Fuzzy -ideals in KU-Algebras

A THESIS Submitted

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Under Supervisions

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## Contents

**Chapter 1 : Basic concepts and results**

<table>
<thead>
<tr>
<th>Subsection</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.1) Introduction to BCK – algebras</td>
<td>2</td>
</tr>
<tr>
<td>1.1 Preliminaries</td>
<td>2</td>
</tr>
<tr>
<td>1.2 Bounded BCK-algebras</td>
<td>4</td>
</tr>
<tr>
<td>1.3 Positive implicative BCK-algebras</td>
<td>5</td>
</tr>
<tr>
<td>1.4 Commutative BCK-algebras</td>
<td>6</td>
</tr>
<tr>
<td>1.5 Implicative BCK-algebras</td>
<td>7</td>
</tr>
<tr>
<td>1.6 BCK-algebras with condition (S)</td>
<td>8</td>
</tr>
<tr>
<td>1.7 The union of two BCK-algebras</td>
<td>9</td>
</tr>
<tr>
<td>1.8 Direct product of BCK-algebras</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Subsection</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.2) BCK ideals</td>
<td>10</td>
</tr>
<tr>
<td>2.1 Definition of BCK –ideal</td>
<td>10</td>
</tr>
<tr>
<td>2.2 Positive implicative ideals</td>
<td>11</td>
</tr>
<tr>
<td>2.3 Implicative ideals</td>
<td>12</td>
</tr>
<tr>
<td>2.4 Commutative ideals</td>
<td>14</td>
</tr>
<tr>
<td>2.5 Maximal ideals</td>
<td>15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Subsection</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.3) Homomorphisms and isomorphisms on BCK-algebras</td>
<td>15</td>
</tr>
<tr>
<td>3.1 Definition of homomorphism on BCK-algebra</td>
<td>16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Subsection</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.4) Some fuzzy concepts in BCI-algebras</td>
<td>17</td>
</tr>
<tr>
<td>4.1 Definition of fuzzy sets</td>
<td>17</td>
</tr>
<tr>
<td>4.2 Definition of fuzzy relations</td>
<td>19</td>
</tr>
</tbody>
</table>
Some types of algebras related to BCK-algebra

5.1 BCI-algebras
5.2 BCH-algebras
5.3 BCC-algebras
5.4 d-algebras
5.5 Q-algebras
5.6 B-algebras
5.7 BM-algebras
5.8 BF-algebras
5.9 BE-algebras
5.10 BG-algebras
5.11 KU-algebras
5.12 CI-algebras
5.13 TM-algebras
5.14 BCL-algebras
5.15 BRK-algebras

Chapter 2: Fuzzy Ideals of KU-Algebras

2.1 Introduction
2.2 Fuzzy KU-ideals of KU-algebras
2.3 Characterization of fuzzy KU-ideal by their level KU-ideals
2.4 Image and Preimage of anti fuzzy KU-ideals under the mapping f
2.5 Cartesian product of fuzzy KU-ideal
Chapter 3: Anti- Fuzzy KU Ideals of KU - Algebras .............................................66

§ 3.1 Anti Fuzzy KU- ideals of KU-algebras.........................................................67
§ 3.2 Characterization of anti fuzzy KU - ideals by their level KU – ideals..........69
§ 3.3 Image and Preimage of anti fuzzy KU – ideals under the mapping f .........70
§ 3.4 Cartesian product of anti fuzzy KU-ideals ....................................................72

References...........................................................................................................75
Appendix.............................................................................................................81
PREFACE

In 1966, Y. Imai and K. Iseki ([29],[30]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras as a generalization of the concept of set-theoretic difference and propositional calculus. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Many concepts related with BCK and BCI-algebras are emerged. For example, the concept of ideals and the concept of filters.

The notion of fuzzy sets was introduced by Zadeh [94] and Rosenfeld [80] applied it to the elementary theory of groups. In 1991, Xi [93] applied the concept of fuzzy set to BCK-algebras. From then on, Jun, Meng et al.[43] and many researchers applied it to the ideal theory of BCK/BCI-algebras.

In 1983, Q.P. Hu and X. Li [22] introduced a wide class of abstract algebras: BCH-algebras. They showed that the class of BCI-algebras is a proper subclass of the class of BCH-algebras and studied a few properties of these algebras. In 1983, Komori ([48]) introduced a notion of BCC algebras, and W. A. Dudek[14] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [16], W. A. Dudek and X. H. Zhang introduced a notion of BCC-ideals in BCC algebras and described connections between such ideals and its congruences. W. A. Dudek and Y. B. Jun ([17], [18],[19]) considered the fuzzification of BCC-ideals in BCC-algebras. They showed that every fuzzy BCC-ideal of a BCC-algebra is a fuzzy BCK-ideal, and showed that the converse is not true by providing an example. They also proved that in a BCC-algebra every fuzzy BCK-ideal is a fuzzy BCC-subalgebra. In 2001, J. Neggers, S.S.Ahn and H.S.Kim [74] introduced a new notion, called a Q-algebra and generalized some theorems discussed in BCI/BCK-algebras. In 2002 ([75]) J. Neggers and H.S.Kim introduced and investigated a class of algebras called B-algebras, which is related to several classes of algebras of interest such as BCH/BCI/BCK-algebras, and which seems to have rather
nice properties without being excessively complicated otherwise,. In 2007, A.Walendziak [90]
introduced a new notion, called a BF-algebra which is a generalization of B-algebra. Ch. B. Kim
and H. S. Kim [50] introduced BG-algebra as a generalization of B-algebra. In 2009 ,
C. Prabpayak and U. Leerawat [78,79] introduced a new algebraic structure which is called KU-
algebras , and studied ideals and congruences in KU-algebras .They gave the concept of
homomorphisms of KU-algebras and investigated some related properties. Moreover they derived
some straightforward consequences of the relations between quotient KU-algebras and
isomorphisms and also investigated some of its properties.

In this thesis ,we introduce two new notions , called fuzzy KU-ideals of KU-algebra (Anti
fuzzy KU-ideals of KU-algebra) , then we study several basic properties which are related to it .
We describe how to deal with the homomorphic image and inverse image of fuzzy (Anti-fuzzy)
KU - ideals . We have also proved that the cartesian product of fuzzy (Anti-fuzzy) KU - ideals in
cartesian product of fuzzy (Anti-fuzzy) KU - algebras are fuzzy (Anti-fuzzy) KU - ideals .

This Thesis consists of three chapters. A short description of each chapter which include
necessary definitions and statements of some major results in that chapter is given below.

Chapter one. Devided into five sections :

First section. Contains definition of concepts BCI /BCK-algebra and discusses its structure ,and
also studies the basic results needed in this thesis .

Section two. Gives the definition of BCK-ideals , some types of ideals of BCK algebra, and
studies some of its different properties .

Section three . concerns with the concept of Fuzzy (subset &relation) and some of its properties
which are needed in this thesis.

Section four .contains the definition of homomorphisms of BCK-algebras and studies some of its
basic properties that are used in this thesis.
Section five. Studies some classes of abstract algebras which related to BCK algebra such as: BCI, BCH, BCC, B, BE,…etc. and shows the relation between them.

Chapter two. In this chapter, we introduce new notion called fuzzy KU-ideals of KU-algebra, study the basic properties which are related to it, describe the homomorphic image and inverse image of fuzzy KU-ideals and finally show that the cartesian product of fuzzy KU-ideals of KU-algebras is fuzzy KU-ideals.

Remark: The results of this chapter is published as a paper in international mathematical forum, vol 6, 2011, no 3139-3149.

Chapter three. In this chapter, we introduce new notion called Anti fuzzy KU-ideals of KU-algebra, study basic properties which are related to it, describe the homomorphic image and inverse image of anti fuzzy KU-ideals, and finally show that the cartesian product of anti fuzzy KU-ideals of KU-algebras is anti fuzzy KU-ideals.

Remark: The results of this chapter is published as a paper in the international journal of algebra and statistics, vol.1, no.1 (2012), 92-99.
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I would like to thank my family for their support, patience and encouragement.
Chapter 1

Basic concepts and results
Chapter (1)
Basic concepts and results

In this introductory chapter, we state the various definitions of the terms and results related to our thesis.

§(1.1) Introduction to BCK – algebras:

In this section, we review some definitions and results that are needed in this thesis. To return to the main results in this section, the reader can refer to ([33],[34],[37],[38],[41],[44],[45],[81],[88]).

1.1 Preliminaries:

The study of BCI (BCK) -algebras was initiated by K. Iseki in 1966 [30], as a generalization of the concept of set-theoretic difference and propositional calculus.

Definition 1.1.1: Let $X$ be a set with a binary operation “$*$” and a constant $0$, then $(X, *, 0)$ is called a BCI-algebra, if it satisfies the following axioms:

\begin{align*}
(BCI -1) & \; ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0, \\
(BCI -2) & \; (x \ast (x \ast y)) \ast y = 0, \\
(BCI -3) & \; x \ast x = 0, \\
(BCI -4) & \; x \ast y = 0 \text{ and } y \ast x = 0 \text{ imply } x = y,
\end{align*}

for all $x, y, z \in X$.

If a BCI-algebra $X$ satisfies the identity $0 \ast x = 0$, for all $x \in X$, then $X$ is called a BCK algebra. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. For brevity $X$ is called a BCK-algebra.

A binary relation $\leq$ in $X$ is defined by: $x \leq y$ if and only if $x \ast y = 0$, then $(X, \ast, 0)$ is a BCK-algebra if and only if it satisfies that:
(BCI '1) : \[(x \ast y) \ast (x \ast z) \leq (z \ast y),\]
(BCI '2) : \[x \ast (x \ast y) \leq y,\]
(BCI '3) : \[x \leq x,\]
(BCI '4) : \[x \leq y, y \leq x \text{ implies } x = y,\]
(BCI '5) : \[0 \leq x,\]
(BCI '6) : \[x \leq y \text{ if and only if } x \ast y = 0.\]

In a BCK-algebra \((X, \ast, 0)\), the following properties are satisfied:

1. \(x \leq y\) implies \(z \ast x \leq z \ast y,\)
2. \(x \leq y\) and \(y \leq z\) imply \(x \leq z,\)
3. \((x \ast y) \ast z = (x \ast z) \ast y\)
4. \((x \ast y) \leq z\) implies \(x \ast z \leq y,\)
5. \((x \ast z) \ast (y \ast z) \leq x \ast y,\)
6. \(x \leq y\) implies \(x \ast z \leq y \ast z,\)
7. \((x \ast (x \ast y)) \ast (y \ast x) \leq x \ast (x \ast (y \ast x)))\)
8. \(x \ast y \leq x,\)
9. \(x \ast 0 = x.\)

**Example 1.1.2**: Let \(X = \{0, 1, 2, 3, 4\}\) in which \(\ast\) is defined by the following table:

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Then \(X\) is BCK-algebra.

**Theorem 1.1.3**: An algebra \((X, \ast, 0)\) of type \((2, 0)\) is a BCK-algebra if and only if it satisfies the following conditions:

- \((BCI_1) : [(x \ast y) \ast (x \ast z)] \ast (z \ast y) = 0,\)
- \((BCI_a) : x \ast (0 \ast y) = x\) and \((BCI_4) : x \ast y = 0 = y \ast x\) implies \(x = y\), for all \(x, y, z \in X,\)
For any \( x,y \) in \( X \) denote \( x \land y = y \ast (y \ast x) \).

Obviously \( x \land x = x \), \( x \land 0 = 0 \land x = 0 \). But in general, \( x \land y \neq y \land x \).

**Definition 1.1.4**: Let \((X, \ast, 0)\) be a BCK-algebra, and let \( S \) be a non-empty subset of \( X \), then \( S \) is called a sub-algebra of \( X \), if for all \( x, y \) in \( S \), \( x \ast y \in S \), i.e \( S \) is closed under the binary operation \( \ast \) of \( X \).

**Example 1.1.5**: Let \( X = \{0,1,2\} \) in which \( \ast \) is defined by the following table:

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It is clear that \( S = \{0,2\} \) is BCK sub-algebra of \( X \).

**Theorem 1.1.6**: Suppose that \((X, \ast, 0)\) is a BCK-algebra and let \( S \) be a sub-algebra of \( X \) then:

(i) \( 0 \in S \),

(ii) \((S, \ast, 0)\) is also a BCK-algebra,

(iii) \( X \) is a subalgebra of \( X \),

(iv) \( \{0\} \) is also a subalgebra of \( X \).

**Theorem 1.1.7**: If we are given a non-zero element \( x_0 \) of a BCK-algebra \((X, \ast, 0)\), then \((\{0, x_0\}, \ast, 0)\) is a sub-algebra of \( X \).

**Lemma 1.1.8**: In a BCK-algebra \((X, \ast, 0)\), for all \( x, y, z \) in \( X \):

(a) If \( x \neq y \) then \( y \ast x \neq 0 \), whenever \( x \ast y = 0 \),

(b) \( x \ast y = z \) implies \( z \ast x = 0 \).

### 1.2 Bounded BCK-algebras [32]:

**Definition 1.2.1**: If there is an element \( 1 \) of BCK-algebra \( X \) satisfying \( x \leq 1 \) for all \( x \) in \( X \), then the element \( 1 \) is called unit of \( X \). A BCK-algebra with unit is said to be bounded.

**Example 1.2.2**: Let \( X = \{0,1,2,3,4\} \) in which \( \ast \) is defined by the following table:
The \( (X, \ast, 0) \) is bounded BCK-algebra with unit 4.

**Note**: In a bounded BCK–algebra \( X \), we denote \( 1 \ast x \) by \( N_x \).

**Definition 1.2.3**: For a bounded BCK–algebra \( X \), if an element \( x \) satisfies \( NN_x = x \), then \( x \) is called an involution.

In a bounded BCK–algebra \( X \), we have:

(a) \( N_1 = 0 \), \( N_0 = 1 \),
(b) \( NN_x \leq x \),
(c) \( N_y \ast N_x \leq y \ast x \),
(d) \( y \leq x \) implies \( N_x \leq N_y \),
(e) \( N_x \ast y = N_y \ast x \),
(f) \( NN_N = N \).

**Theorem 1.2.4**: In a bounded BCK–algebra \( X \), we have \( x \ast N_y = y \ast N_x \), for all \( x \) and \( y \in S (X) \) where \( S (X) \) is the set of all involutions of a bounded BCK–algebra.

Note that: \( NN_0 = N_1 = 0 \), and \( NN_1 = N_0 = 1 \), then the elements 0 and 1 are contained in \( S (X) \). Hence \( S (X) \) is non-empty.

**Theorem 1.2.5**: For any bounded BCK–algebra \( X \), we have \( S(X) \) is a bounded sub-algebra of \( X \).

**Theorem 1.2.6[21]**: A BCI–algebra \( X \) satisfying \( x \ast (y \ast z) = (x \ast y) \ast z \) is a group in which every element is an involution.

### 1.3 Positive implicative BCK-algebras [36]:

**Definition 1.3.1**: A BCK–algebra \( X \) is said to be positive implicative if it satisfies:

\[
\forall \ x, y \text{ and } z \in X, (x \ast z) \ast (y \ast z) = (x \ast y) \ast z.
\]

**Example 1.3.2**: Let \( X = \{0,1,2,3\} \) in which \( \ast \) is defined by the following table:
Then (X, *, 0) is a positive implicative BCK-algebra.

**Theorem 1.3.3**: Let (X, *, 0) be a BCK-algebra, then the following conditions are equivalent:

a) X is a positive implicative,
b) x·y=(x·y)·y,
c) (x·(x·y))·(y·x)=x·(x·(y·x)),
d) x·y=(x·y)·(x·(x·y)),
e) x·(x·y)=(x·(x·y))·(x·y),
f) (x·(x·y))·(y·x)=(y·(y·x))·(x·y).

**Theorem 1.3.4**: An algebra (X, *, 0) of type (2.0) is positive implicative BCK-algebra if and only if it satisfies BCI-1, BCK-5, (f) (x·(x·y))·(y·x)=(y·(y·x))·(x·y) and (g) x·0=x.

**Theorem 1.3.5**: Let (X, *, 0) be a BCK-algebra, then the following conditions are equivalent:

a) X is a positive implicative,
b) (x·y)·z=0 implies (x·z)·(y·z)=0,
c) (x·y)·y=0 implies x·y=0.

**1.4 Commutative BCK-algebras ([86],[87]):**

**Definition 1.4.1**: A BCK-algebra (X, *, 0) is said to be commutative if it satisfies:

∀ x, y ∈ X, x·(x·y) = y·(y·x). i.e x·y = y·x.

**Example 1.4.2**: Let X = {0,1,2,3} in which * is defined by the following table:

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X is a positive implicative BCK-algebra.
Then \(( X, \ast, 0 )\) is a commutative BCK-algebra.

**Theorem 1.4.3**: For a BCK-algebra \(X\), the following conditions are equivalent:

a) \(X\) is commutative,

b) \(x \ast (x \ast y) \leq y \ast (y \ast x)\),

c) \((x \ast (x \ast y)) \ast (y \ast (y \ast x)) = 0\).

**Definition 1.4.4**: For a BCK-algebra \(X\), the set \(A(x) = \{y \in X : y \leq x\}\) is called an initial section of an element \(x\).

**Theorem 1.4.5**: A BCK-algebra \(X\) is commutative if and only if:

\[
A(x) \cap A(y) = A(x \land y), \text{ for all } x, y \text{ in } X.
\]

**Theorem 1.4.6**: A BCK-algebra \(X\) is commutative if and only if \(x \ast (x \ast y) = y \ast (y \ast (x \ast (x \ast y)))\), for all \(x, y\) in \(X\).

### 1.5 Implicative BCK-algebras ([31],[36]):

**Definition 1.5.1**: A BCK-algebra \((X, \ast, 0)\) is said to be implicative if it satisfies:

\[
\forall x, y \in X, \quad x = x \ast (y \ast x).
\]

**Example 1.5.2**: Let \(X = \{0,1,2\}\) in which \(\ast\) is defined by the following table:

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Then \(( X, \ast, 0 )\) is implicative BCK-algebra.