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A General Quantum Difference Operator and its Calculus

A THESIS

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By

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DEDICATION

This thesis is dedicated to my family.

For their endless love, support and encouragement.

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Contents

Publications related to this thesis.

1. Nashat Faried, Enas M. Shehata and Rasha M. El Zafarani, On homogenous second order linear general quantum difference equations, Journal of Inequalities and Applications, Springer, 2017:198, (2017). DOI 10.1186/s13660-017-1471-3.
2. Nashat Faried, Enas M. Shehata and Rasha M. El Zafarani, Theory of nth-order linear general quantum difference equations, Submitted.
3. Nashat Faried, Labib R. Awad, Enas M. Shehata and Rasha M. El Zafarani, Systems of linear general quantum difference equations, Submitted.

Notations

- $\mathbb{R}$: the set of real numbers.
- $\mathbb{N} := \{1, 2, \dots\}$.
- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- $\mathbb{C}$: the set of complex numbers.
- $\mathbb{X}$: is a Banach space.
- $\|.\|$: the norm defined on $\mathbb{X}$.
- I: is an interval subset of $\mathbb{R}$.
- J: is a subinterval of I containing the unique fixed point s_0 of the function β.
- $\beta^k(t) := \underbrace{\beta o \beta o \dots o \beta}_{k-times}(t), \beta^0(t) = t$.
- $\acute{f}(t)$: is the usual derivative of the function f at the point t.
- $S(y_0, b) := \{y \in \mathbb{X} : \|y - y_0\| \leq b\}$.
- $R := \{(t, y) \in I \times \mathbb{X} : |t - s_0| \leq a, \|y - y_0\| \leq b\}$ is a rectangle, where a, b are fixed positive real numbers.
- $M := sup_{(t,y)\in R} \|f(t, y)\| < \infty$.
- $D_\beta^n f := D_\beta(D_\beta^{n-1} f)$; $n \in \mathbb{N}_0$, where f is β-differentiable n times over I, and $D_\beta^0 f = f$.

Abstract

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A General Quantum Difference Operator and its Calculus

By

Rasha Mohamady El-Zafarani

The general quantum difference operator, D_β , is defined by

$$D_\beta f(t) = \begin{cases} \dfrac{f\,(\beta(t))- f\,(t)}{\beta(t)- t}, & t \neq s_0, \\ f'(s_0), & t = s_0, \end{cases}$$

where $f: I \to \mathbb{X}$ is a function defined on an interval $I \subseteq \mathbb{R}$ and $\beta : I \to I$ is a strictly increasing continuous function defined on I which has only one fixed point $s_0 \in I$ and satisfies the inequality: $(t - s_0)(\beta(t) - t) \leq 0 \; for\ all\ t \in I.$

In this thesis, we prove the existence and uniqueness of solutions of the β-Cauchy problem of second order β-difference equations

$$a_0(t)D_\beta^2 y(t) + a_1(t)D_\beta y(t) + a_2(t)y(t) = b(t), \qquad t \in I,$$

$a_0(t) \neq 0$, in a neighborhood of the unique fixed point s_0 of the function β. We also construct a fundamental set of solutions for the second order linear homogeneous β-difference equations when the coefficients are constants and study the different cases of the roots of their characteristic equations. Then we drive the Euler-Cauchy β-difference equation. Furthermore, we give the sufficient conditions for the existence and uniqueness of solutions of the β-Cauchy problem of β-difference equations. We also establish the fundamental set of solutions when the coefficients are constants, the β-Wronskian associated with D_β, and Liouville's formula for β-difference equations. In addition, we deduce the undetermined coefficients, the variation of parameters and the annihilator methods for the non-homogeneous β-difference equations. Finally, we introduce the solutions of homogenous and non-homogenous systems of linear β-difference equations.

Keywords: A general quantum difference operator; Linear general quantum

difference equations; Euler-Cauchy general quantum difference equation; β-Wronskian.

Introduction

Quantum calculus is known as the calculus without limits. It substitutes the classical derivative by a difference operator which allows to deal with sets of non-differentiable functions. Quantum difference operators have an interesting role due to their applications in several mathematical areas such as the calculus of variations, orthogonal polynomials, basic hyper-geometric functions, economical problems with a dynamic nature, quantum mechanics and the theory of scale relativity; see, e.g. [1, 6, 9, 10, 12, 20, 21, 22, 23, 25].

The general quantum difference operator, D_β, is defined, in [15, 24, Chapter 2], by

$$D_\beta f(t) = \begin{cases} \dfrac{f\,(\beta(t)) - f\,(t)}{\beta(t) - t}, & t \neq s_0, \\ f'(s_0), & t = s_0, \end{cases}$$

where $f\colon I \to \mathbb{X}$ is a function defined on an interval $I \subseteq \mathbb{R}$ and $\beta : I \to I$ is a strictly increasing continuous function defined on I which has only one fixed point $s_0 \in I$ and satisfies the inequality: $(t - s_0)(\beta(t) - t) \leq 0 \; for\; all\; t \in I$. The function f is said to be β-differentiable on I, if the ordinary derivative f' exists at s_0.

Jackson and Hahn difference operators are special linear forms of the general difference operator, D_β. The Jackson q-difference operator is defined by

$$D_q f(t) = \frac{f\,(qt\,) - f\,(t)}{t(q - 1)}, \qquad t \neq 0,$$

and $D_q f(0) = f'(0)$, where $q \in (0, 1)$, is a fixed number. The function f is

defined on a q −geometric set $\mathbb{A} \subseteq \mathbb{R}$(or $\mathbb{C}$) such that whenever $t \in \mathbb{A}$, $qt \in \mathbb{A}$.

Also, the Hahn difference operator which is a tool for constructing families of orthogonal polynomials is defined by

$$D_{q,\omega} f\,(t) = \frac{f\,(qt + \omega) - f\,(t)}{t(q - 1) + \omega}, \qquad t \neq \omega_0,$$

where $q \in (0, 1)$, $\omega > 0$ are fixed and $\omega_0 = \frac{\omega}{1-q}$. See [2, 4, 5, 8, 14, 19].

In [15, 24, Chapter 2], the definition of the β-derivative, the β-integral, the fundamental theorem of β-calculus, the chain rule, Leibniz's formula and the mean value theorem were introduced. In [16], the β-exponential, β-trigonometric and β-hyperbolic functions were presented. Some inequalities based on the general quantum difference operator, D_β, such as β-Hölder, β-Minkowski, β-Gronwall, and β-Bernoulli inequalities were presented in [18]. In [17], the existence and uniqueness of solutions of the β-initial value problem

$$D_\beta y(t) = f(t,y), \quad y(s_0) = y_0, \qquad t \in I,$$

associated with D_β were established. In addition, an expansion form for the β-exponential function was deduced by using the successive approximations method. This thesis is devoted to proceed the calculus based on the general quantum difference operator, D_β.

The thesis consists of five chapters. We summarize the main aims of each chapter as follows.

In **Chapter 1**, we present the basic results about the general quantum difference operator and the associated calculus. In **Chapter 2**, we prove the existence and uniqueness of solutions of the β-Cauchy problem of second order β-difference equations

$$a_0(t)D_\beta^2 y(t) + a_1(t)D_\beta y(t) + a_2(t)y(t) = b(t), \qquad t \in I,$$

$a_0(t) \neq 0$, in a neighborhood of the unique fixed point s_0 of the function β. These equations are based on the general quantum difference operator D_β. We also construct a fundamental set of solutions for the second order linear homogeneous β-difference equations when the coefficients are constants and study the different cases of the roots of their characteristic equations. Finally, we drive the Euler-Cauchy β-difference equation. The results of this chapter were published in : "Journal of Inequalities and Applications, 2017:198, (2017). DOI 10.1186/s13660-017-1471-3", entitled "On homogenous second order linear general quantum difference equations". In **Chapter 3**, we give the sufficient conditions for the existence and uniqueness of solutions of the β-Cauchy problem of β-difference equations. Furthermore, we present the fundamental set of solutions when the coefficients are constant, the β-Wronskian associated with D_β and the Liouville's formula for β-difference equations. Finally, we introduce the undetermined coefficients, the variation of parameters and the annihilator methods for the non-homogeneous β-difference equations. In **Chapter 4**, we present the solutions of homogenous and non-homogenous systems of linear β-difference equations. In **Chapter 5**, we give a conclusion of our thesis and a future work.

Chapter 1

General Quantum Difference Calculus

Chapter 1

General Quantum Difference Calculus

The general quantum calculus is based on the general quantum difference operator

D_β, which is defined, in [15, 24, Chapter 2], by:

$$D_\beta f(t) = \begin{cases} \frac{f\,(\beta(t)) - f\,(t)}{\beta(t) - t}, & t \neq s_0, \\ \acute{f}(s_0), & t = s_0, \end{cases}$$

where $f : I \to \mathbb{X}$ is a function defined on an interval $I \subseteq \mathbb{R}$, $\mathbb{X}$ is a Banach space and $\beta : I \to I$ is a strictly increasing continuous function defined on I. The function β has only one fixed point $s_0 \in I$ and satisfies the inequality: $(t - s_0)(\beta(t) - t) \leq 0 \; for\ all\ t \in I$. The function f is said to be β-differentiable on I, if the ordinary derivative $\acute{f}$ exists at s_0. The β-difference operator yields the Hahn difference operator when $\beta(t) = qt + \omega, \omega > 0$, $q \in (0,1)$, the Jackson q-difference operator when $\beta(t) = qt$, $q \in (0,1)$, the forward difference operator when $\beta(t) = t + \omega, \omega > 0$, the n, q-power quantum difference, $D_{n,q}$, when $\beta(t) = qt^n$, $n \in 2\mathbb{N} + 1$, $q \in (0,1)$, and the difference operator a, b when $\beta(t) = at + b$, with $a \geq 1, b \geq 0$ and $a + b > 1$. See [2, 3, 4, 5, 7, 8, 19]. In this chapter, we present an overview of the calculus associated with the general quantum difference operator, D_β.

1.1 $\boldsymbol{\beta}$-Differentiation and $\boldsymbol{\beta}$-Integration

In this section, we present some basic results of β-differentiation and β-integration from [15, 24, Chapter 2].

1.1.1 $\boldsymbol{\beta}$-Differentiation

We have a strictly increasing continuous function defined on , that has only one fixed point $s_0 \in I$ and satisfies the following inequality :

$$(t - s_0)(\beta(t) - t) \leq 0 \qquad for\ all \quad t \in I,$$

where the equality holds only if $t = s_0$.

Lemma 1.1.1. [15] The following statements are true:

(i) The sequence of functions $\{\beta^k(t)\}_{k=0}^{\infty}$ converges uniformly to the constant function $\hat{\beta}(t) := s_0$ on every compact interval $V \subseteq I$ containing s_0.

(ii) The series $\sum_{k=0}^{\infty} |\beta^k(t) - \beta^{k+1}(t)|$is uniformly convergent to $|t - s_0|$on every compact interval $V \subseteq I$ containing s_0.

Lemma 1.1.2. [15] If $f : I \to \mathbb{X}$ is continuous function at s_0, then the sequence $\{f(\beta^k(t))\}_{k\in \mathbb{N}_0}$converges uniformly to $f(s_0)$ on every compact interval $V \subseteq I$ containing s_0.

Theorem 1.1.3. [15] If $f : I \to \mathbb{X}$ is continuous at s_0, then the series $\sum_{k=0}^{\infty} \left\| (\beta^k(t) - \beta^{k+1}(t)) \times f(\beta^k(t)) \right\|$ is uniformly convergent on every compact interval $V \subseteq I$ containing s_0.

Remark 1.1.4. [15] The properties of the β-difference operator is

(i) D_β is a linear operator.

(ii) If f is β-differentiable at t, then $f(\beta(t)) = f(t) + (\beta(t) - t)D_\beta f(t)$.

(iii) If f is β-differentiable, then f is continuous at s_0.

Theorem 1.1.5. [15] Assume that $f : I \to \mathbb{X}$ *and* $g : I \to \mathbb{R}$ are β-differentiable at $t \in I$. Then:

(i) The product $fg : I \to \mathbb{X}$ is β-differentiable at t and

$$\begin{aligned} D_\beta (fg)(t) &= (D_\beta f(t))g(t) + f(\beta(t))D_\beta g(t) \\ &= \left(D_\beta f(t)\right) g\left(\beta(t)\right) + f(t)D_\beta g(t), \end{aligned}$$

(ii) f/g is β differentiable at t and

$$D_\beta \left(\frac{f}{g}\right)(t) = \frac{\left(D_\beta f(t)\right) g(t) - f(t)D_\beta g(t)}{g(t)g\left(\beta(t)\right)},$$

provided that $g(t)g(\beta(t)) \neq 0$.

Example 1.1.6. [15]

1. $D_\beta t^n = \sum_{k=0}^{n-1} \left(\beta(t)\right)^{n-k-1} t^k, \; t \in I, \; n \geq 1.$

2. For $t \neq 0, D_\beta = -\frac{1}{t\beta(t)}, t \in I, \quad \beta(t) \neq 0$

3. If $f : I \to \mathbb{R}^2$ defined by $f(t) = (t^2, 2t)$ and $\beta(t) = \frac{1}{2}t + 1$, then

$$D_\beta f(t) = \frac{\left(-\frac{3}{4}t^2 + t + 1, 2 - t\right)}{1 - \frac{1}{2}t}.$$