ON DECOMPOSABLE OPERATORS

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الله

بسم الله الرحمن الرحيم

وقل اعملوا فسيرى الله عملكم ورسوله والمؤمنون

صدق الله العظيم

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INTRODUCTION

Introducing the concept of spectral operators in 1954, N. Dunford was proposing a process of extending the spectral theory of self-adjoint operators on Hilbert spaces to ever more general operators on Banach spaces. His new theory depends on some algebraic and topological structures outside the domain of such spectral operators.

A class of operators with a well-developed spectral theory was introduced by C. Foias [19], 1963, under the name of "decomposable operators". Such spectral decomposition is defined only in regard to the operators invariant subspaces. In this way, the spectral theory can be conceived as an axiomatic system functioning within the underlying Banach space with possible extensions to more general topological spaces.

Now we give a comparison and the relation between the definitions of spectral operators in the sense of N. Dunford, and the decomposable operators in the sense of C. Foias.

Let X be a complex Banach space, B(X) the Banach algebra of the linear bounded operators on X, P_X the set of the projections of X, and $\mathcal{B}$ the family of the Borel subsets of the complex plane $\mathbb{C}$.

A mapping $E:\mathcal{B} \to P_X$ is called a spectral measure if:

(I) $E(B_1 \cap B_2) = E(B_1)E(B_2), B_1, B_2 \in \mathcal{B}$;

(II) $E(\bigcup_{n=1}^{\infty} B_n)x = \sum_{n=1}^{\infty} E(B_n)x,\ B_n \in \mathcal{B},\ B_n \cap B_m = \phi$ if $n \neq m$, $x \in X$;

(III) $E(\mathbb{C}) = I$.

N. Dunford defined the spectral operators as follows:

$T\in B(X)$ is called spectral operator if there exists a spectral measure E such that

(IV) $TE(B) = E(B)T$, $B\in\mathcal{B}$;

(V) $\sigma(T/E(B)X) \subset \bar{B}$, $B\in\mathcal{B}$.

The spectral measure E verifying (IV) and (V) is uniquely determined by T, and it is called the spectral measure of T.

Let $T\in B(X)$ be a spectral operator with spectral measure E, and let $F\subset\mathbb{C}$ be a closed set. Then, (by [12], xv, th. 4),

$E(F)X = \{ x: x\in X$ and $\sigma_T(x) \subset F \} \equiv X_T(F)$,

where $\sigma_T(x)$ is the local spectrum of T at x.

C. Apostol [5], 1968, proposed the following generalization for the notion of spectral measure, namely, "the spectral capacity": Let $S(X)$ be the family of all closed subspaces of the Banach space X, and $\mathfrak{F}$ be the family of all closed subsets of the complex plane $\mathbb{C}$. A spectral capacity Ξ is a mapping $\Xi: \mathfrak{F} \to S(X)$ with the properties:

(i) $\Xi(\phi) = \{0\}$, $\Xi(\mathbb{C}) = X$;

(ii) $\bigcap_{n=1}^{\infty} \Xi(F_n) = \Xi(\bigcap_{n=1}^{\infty} F_n)$, $F_n\in\mathfrak{F}$;

(iii) if $\{G_i\}_{i=1}^{n}$ is an open covering of $\mathbb{C}$ then $X = \sum_{i=1}^{n} \Xi(\bar{G}_i)$.

An operator $T\in B(X)$ has a specrtal capacity Ξ if for every $F\in\mathfrak{F}$ we have

(iv) $T\,\Xi(F) \subset \Xi(F)$;

(v) $\sigma(T/\Xi(F)) \subset F$.

It is evident that a spectral operator T with spectral measure E has a spectral capacity Ξ , where

$$\Xi(F) = E(F)X \quad \text{for} \quad F \in \mathfrak{F} \quad (\text{see } [10]).$$

Following C. Foias [19] we say that $T \in B(X)$ is decomposable if, for every finite open covering $\{G_i\}_{i=1}^{n}$ of $\sigma(T)$, there exists a system $\{Y_i\}_{i=1}^{n}$ of spectral maximal spaces of T such that:

$$\sigma(T/Y_i) \subset G_i \ , \quad 1 \leqslant i \leqslant n \ ;$$

$$X = \sum_{i=1}^{n} Y_i \ .$$

For a decomposable operator $T \in B(X)$, if the mapping $\Xi : \mathfrak{F} \to S(X)$ is defined by $\Xi(F) = X_T(F)$, then Ξ is a spectral capacity of T. C. Foias [20], 1968, proved that if $T \in B(X)$ has a spectral capacity Ξ , then T is a decomposable operator and

$$\Xi(F) = X_T(F) \quad \text{for every} \quad F \in \mathfrak{F}. \qquad \ldots (*)$$

Moreover, (*) shows that the spectral capacity Ξ is uniquely determined by T.

In this manner one obtains a new definition of the decomposable operators as it is the definition of the spectral operators in Dunford's sense, since the only difference is to replace the spectral measures by spectral capacities.

C. Apostol [3] investigated the following results for T/Y and $\hat{T}$ in the case that Y is a spectral maximal space of $T \in B(X)$ and T is (strongly) decomposable (see the definition in chapter two) where T/Y and $\hat{T}$ are the restriction and the quotient operators, respectively:

1- If $T \in B(X)$ is a strongly decomposable operator, then for any spectral maximal space Y of T the operator T/Y is strongly decomposable.

2- The operator $T \in B(X)$ is strongly decomposable iff T/Y is decomposable for any spectral maximal space Y of T.

3- Let $T \in B(X)$ be a strongly decomposable operator. Then for any spectral maximal space Y of T, the operator $\hat{T}$ is strongly decomposable.

4- If $T \in B(X)$ is decomposable and Y is a spectral maximal space of T, then we have $\sigma(\hat{T}) = \overline{\sigma(T) \setminus \sigma(T/Y)}$.

I. Colojoara; C. Foias [9] and C. Apostol [4,5] proved the following results for the stability of the spectral decompositions under the direct sums and the functional calculus:

5- If $T_i \in B(X_i)$, $i=1,2$, are decomposable operators. Then $T_1 \oplus T_2 \in B(X_1 \oplus X_2)$ is also decomposable.

6- Let $T \in B(X)$ be (strongly) decomposable and $P \in B(X)$ be a projection commuting with T. Then T/PX is a (strongly) decomposable operator.

7- Let $T \in B(X)$ be decomposable, and let $f : D \to \mathbb{C}$ be analytic on an open neighborhood D of $\sigma(T)$. Then $f(T)$ is decomposable.

8- Let $T \in B(X)$, and let $f : D \to \mathbb{C}$ be an analytic and injective function on an open neighborhood D of $\sigma(T)$. If $f(T)$ is decomposable then T is decomposable.

In 1969, F. H. Vasilescu [43] constructed a certain spectral theory for closed linear operators on a Banach space (class of S-residually decomposable operators). These operators have a suitable spectral behaviour on subsets of their spectra after eliminating some residual part.

A family of open sets $\{G_i\}_{i=1}^{n} \cup \{G_s\}$ is an S-covering of the closed set $\sigma \subset \mathbb{C}_\infty$ ($\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$) if :

$$S \cup \sigma \subset \bigcup_{i=1}^{n} G_i \cup G_s ;$$

$$\bar{G}_i \cap S = \phi , \ 1 \leq i \leq n.$$

F. H. Vasilescu [43] defined the S-residually decomposable operators as follows:

Let C(X) be the class of the closed linear operators in a Banach space X.

An operator $T \in C(X)$ is called S-residually decomposable if :

(α) For any closed set $F \subset \mathbb{C}_\infty$ with $F \cap S = \phi$ the family

$I_{T,F} = \{Y : T/Y \in B(Y) \text{ and } \sigma(T/Y) \subset F \}$

is directed and has a maximal element $X_{T,F}$.

(β) For any S-covering $\{G_i\}_{i=1}^{n} \cup \{G_s\}$ of $\sigma(T)$, where $S \subset \sigma(T)$, there exist the invariant subspaces $\{Y_i\}_{i=1}^{n}$ of T with the properties:

(β_1) $Y_i \subset D_T$, $1 \leq i \leq n$, where D_T is the domain of T;

(β_2) $\sigma(T/Y_i) \subset G_i$, $1 \leq i \leq n$;

(β_3) every $x \in X$ has a decomposition of the form:

$$x = x_1 + x_2 + \dots + x_n + x_s$$

where $x_i \in Y_i (1 \leq i \leq n)$ and $\sigma_T(x_s) \subset \bar{G}_s$.

The theory of S-decomposable operators, more general than that of the decomposable operators, was first studied in the bounded case by I. Bacalu [6], 1975, and was extended to the case of a closed operator by B. Nagy [35], 1980. Loosely speaking, an operator is S-decomposable if it shows a good spectral behaviour (connected with decomposability in the sense of C. Foias) outside a certain subset, denoted by S, of its spectrum.

I. Bacalu [6,7] proved the following results:

9- Let $T \in B(X)$ be a strongly S-decomposable operator and Y be a spectral maximal space of T. Then T/Y is an S_1-decomposable operator, where

$$S_1 = S \cap \sigma(T/Y) .$$

10- Let $T_i \in B(X_i)$, i=1,2, and $S=S_1 \cup S_2$, then T_i are S_i-decomposable, i=1,2, iff $T_1 \oplus T_2 \in B(X_1 \oplus X_2)$ is S-decomposable.

11- Let $T \in B(X)$ be S-decomposable and $P \in B(X)$ be a projection commuting with T. Then T/PX is S_1-decomposable, where

$$S_1 = S \cap \sigma(T/PX).$$

W. Shengwang; G. Liu [41] proved the following results:

12- $T \in B(X)$ is S-decomposable iff for every open $G \subset \mathbb{C}$ with $\bar{G} \cap S = \phi$, there is a ν-space Y of T(see the definition in chapter one) such that $\sigma(T/Y) \subset \bar{G}$ and $\sigma(\hat{T}) \subset G^c$.

13- $T \in B(X)$ is S-decomposable iff T^* is S-decomposable .

B. Nagy [35] defined the closed S-decomposable operators as follows:

Suppose $T \in C(X)$ and that the closed set S is contained in $\sigma(T)$.

T is called S-decomposable if, for any open S-covering $\{G_i\}_{i=1}^n \cup \{G_s\}$ of $\sigma(T)$, there exist spectral maximal spaces $\{Y_i\}_{i=1}^n \cup \{Y_s\}$ of T with $Y_i \subset D_T (1 \leqslant i \leqslant n)$ and $Y_s \subset X$ such that

(i) $\sigma(T/Y_i) \subset G_i$ $(1 \leqslant i \leqslant n)$ and $\sigma(T/Y_s) \subset G_s$;

(ii) $X = Y_1 + Y_2 + \ldots + Y_n + Y_s$.

Observe that if $T \in C(X)$ is an S-decomposable operator and S is bounded, then $T \in B(X)$.

W. Shengwang; I. Erdelyi [40], 1984, extended the theory of bounded decomposable operators to the case of the closed operators as follows:

An open set $G \subset \mathbb{C}$ is said to be a neighborhood of ∞, in symbols $G \in V_\infty$, if for $r > 0$ sufficiently large, $\{\lambda \in \mathbb{C} : |\lambda| > r\} \subset G$; i.e. G^c is compact.

$T \in C(X)$ is said to be decomposable if, for any covering $\{G_i\}_{i=0}^n$ of $\sigma(T)$ with $G_o \in V_\infty$, there is a system $\{Y_i\}_{i=0}^n$ of spectral maximal spaces of T such that:

(i) $Y_i \subset D_T$ if G_i is relatively compact in $\mathbb{C}$ $(1 \leqslant i \leqslant n)$;

(ii) $X = \sum_{i=0}^{n} Y_i$ and $\sigma(T/Y_i) \subset G_i$ $(0 \leqslant i \leqslant n)$.

If $\{Y_i\}_{i=0}^{n}$ is a system of invariant subspaces of T then it said that $T\epsilon C(X)$ has the SDP.

The properties of closed operators with SDP received a systematic treatment by I. Erdelyi; W. Shengwang [15,16,40] . They obtained many significant results, one of these results is:

14- $T\epsilon C(X)$ has the SDP iff T^* has the SDP.

The aim of this thesis is:

(i) to study the stability for (strongly) decomposable operators and S-residually decomposable operators under the similarity;

(ii) to investigate some results, analogous to the results 5,6,10 and 11, in the case of the S-residually decomposable operators;

(iii) to investigate some results, analogous to the results 7 and 8, in the case of the S-residually decomposable operators and the S-decomposable operators ;

(iv) to investigate a result, analogous to the result 12, in the case of the closed S-decomposable operators;

(v) to study and establish a duality theorem in the case of the spectral S-decompositions of closed operators.

(vi) to investigate some results, analogous to the results 1,2,3,4 and 9, in the case of the closed S-decomposable operators.

The thesis consists of three chapters:

In chapter one, we state definitions, notions and results (without proofs) needed in the proof of our results in the following chapters.

We state the notions of invariant subspaces, the single-valued extension property, the ν-spaces, the T-absorbing subspaces, the spectral maximal spaces and the functional calculus for an operator T.

Chapter two deals with the decomposable operators, the S-residually decomposable operators and the S-decomposable operators, in the bounded case.

In this chapter, we present the stability of these decompositions under the similarity, the direct sum and the functional calculus.

We have stated and proved theorems 5.4,7.3,7.4,7.6,7.8,7.9,8.3 and 8.5 which show that the required stabilities are satisfied.

Chapter three is devoted to study some recent properties of closed S-decomposable operators. We have extended some results of bounded decomposable and S-decomposable operators to the class of closed S-decomposable operators.

At the same time, we have given some results analogous to that for closed operators with SDP.

In this manner, we have concluded an equivalent definition of the S-decomposability for closed operators, established a duality theorem for closed S-decomposable operators and finally, studied the properties of the restriction and the quotient operators of closed S-decomposable operators.

CONTENTS

SYMBOLS

$\mathbb{C}$	, the complex plane.
$\mathbb{C}_\infty$	, the compactification of $\mathbb{C}$,i.e. $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.
$\bar{A}$	, the closure of a set A.
A^c	, the complement of A(in a given total set).
∂A	, the boundary of A with respect to some topological space.
	For a linear operator T on a Banach space X:
$B(X)$	, the Banach algebra of bounded linear operators on X.
$C(X)$	, the class of all closed operators on X.
D_T	, the domain of T.
$\sigma(T)$	, the spectrum of T.
$\rho(T)$	, the resolvent set.
$R(.;T)$	, the resolvent operator.
$\sigma_a(T)$	, the approximate point spectrum.
$\sigma_p(T)$	, the point spectrum.
$\sigma_{com}(T)$	, the compression spectrum.
$\sigma_r(T)$	, the residual spectrum.
$\sigma_c(T)$	, the continuous spectrum.
S_T	, definition p. 5.
$\nu_T(x)$	, definition p. 5.
$\sigma_T(x)$	, the local spectrum, p. 5.
$\rho_T(x)$	, the local resolvent set, p. 5.
$\tilde{x}_T$	, the maximal analytic extension of $R(.,T)x, x \in X$, p.6.
$X_T(F)$	, definition p. 6.
$X_{T,F}$	, definition p. 7.