Game dynamics of the simultaneous and alternating Prisoner's Dilemma and other iterated games

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Summary

This work deals with a recent branch of game theory, namely evolutionary game dynamics. This field has been created by J.Maynard Smith [16], E.C Zeeman [42] and others, and it studies the interaction of players adopting to each other's strategies.

The aim of this work is to study the evolution of the behaviour of the stochastic strategies (with a certain probability for each decision) in the iterated 2x2-games considering the game as a dynamical system. We take in our analysis the Prisoner's Dilemma and Chicken games as examples. We deal with two models in our study, simultaneous games and alternating games

In some cases, it is easy to obtain the best strategy (that has highest payoff) if the game is not repeated , but this is different if the game is repeated. Generally there is no best strategy. The evolution theory which we used in our work is a good way to study the behaviour of the strategies for the iterated games.

This work consists of three chapters, in the first and second chapters we study the simultaneous model and in the third chapter we study the alternating model . All models which we shall discuss are symmetric i.e games with symmetric payoff matrix .

In the first chapter we are interested in reactive strategies, where the decisions of the players depend on the opponent's move in the previous round. In this chapter we use the adaptive dynamic (optimal directional change) to study the evolution of the reactive strategies. This study deals with two cases, the case when the probability to repeat the game equal one. In this case we get a new results for the optimal direction in the case of Chicken game. The case of Prisoner's Dilemma was studied in [28]. We also discussed the

case when the probability of repeating the game is less than one.

In the second chapter we considered the strategies corresponding to all possible outcomes defined by two-state automata. Also in this chapter we study the effect of the errors due to perception and the errors in implementation. We assume a small noise level for each error and we describe the methods which compute the limit payoff values when the noise levels tends to zero. The limiting payoff matrix which we get is used to study the behaviour of the strategies. All the results of this chapter are gathered in two papers [43], [44]. The paper [43] is accepted for publication in the journal of mathematical biology. the paper [44] is submitted for publication.

In the third chapter we study the evolution of strategies in the alternating games. We study the two cases of this model, namely the strictly alternating game and the randomly alternating game . In this chapter we use the same techniques of chapters one and two. The results of this chapter considered as a generalization in the case of alternating game.

Introduction

Game theory is not a prescriptive way of how to play a game. Rather it is a set of ideas and techniques for analysis these mathematical models. It doesn't tell you how to play the game, but describes properties that certain ways of playing the game have, and which you might think desirable. Even when the analysis suggests a best way of playing the game, it only does it assuming that everyone is playing in the 'best way' they can. It never allows for ways of punishing your opponent if he makes a mistake, which is the way most players do.

In this section we shall introduce the basic concepts and terminology of the game theory.

In the theory of game we always have at least two participants. These participants are called players labelled by I, II, etc. Suppose that we have two players I and II and a set of choices I_1, I_2, ..., I_n for each player. These choices are called moves or options. We have two types of games: alternating games and simultaneous games. In the alternating game one of the players (first player) choses one of the options I_1, I_2, ..., I_n then the other player (second player) choses one of these options after knowing the option of the first player. In the case of alternating games we must indicate the player who will start the game. In the simultaneous game, both players chose their options in the same time without knowing the choice of the other player. The process of choosing one option of each player is called one round. If the game consists of more than one round, then the game is called a repeated or iterated game. The game can be finite or infinite according to the number repetition of the game. After each round, each player gets a certain reward. If player I chooses I_i and player II chooses I_j , then the reward of player I will be a_{ij} , while the reward of player II will be a_{ji} . In the case when $a_{ij} = a_{ji}$, we say that the game is symmetric. The elements of rewards are written in a matrix A_{ij} of order nxn which is called the payoff matrix of the game. At the end of the game, each player receive a payoff.. We will always assume that the payoff is given by a real number. The game is called zero-sum game if the sum of the player's payoff is zero, otherwise it is called non-zero sum game.

In some games, the analysis of the game leads to a specific strategy for each player. A strategy for a player is a description of the decisions he will make at all the possible situations that can arise in the game.

Normal form:

The way of analysing the games starts by listing all possible strategies for each player, i.e strategies $e_1, e_2, \ldots, e_n$ for player I and stratcgies $f_1, f_2, \ldots, f_m$ for player II . If player I is playing strategy e_i and player II is playing strategy f_j , then player I gets payoff e_{ij} and player II gets payoff f_{ji} .The outcome in this case is (e_{ij}, f_{ji}) . This is the normal form of the game (asymmetric game or two-population game) . The payoff values of the game can be recorded in nxm matrix which contain the pairs (e_{ij}, f_{ji}) as elements. This matrix is called the payoff matrix.

If there are a chance moves in the game , then when player I plays e_i against player II who plays f_j , the payoff depends on the outcome of the chance move .So we multiply each payoff by the probability of the chance event that will give rise to it , and add all these products together . This gives the average or expected payoff.

In some games player II has the same possible strategies $e_1, e_2, \ldots, e_n$ as player I and $e_{ij}=e_{ji}$. In this case the game is described by the symmetric payoff matrix (symmetric game or one population game).

We note that our study in this work concerns the repeated symmetric 2x2-games.

Evolutionarily stable strategy (ESS) :

Let us consider a game with two players I, II and two options I_1, I_2 for each player. Each player think about how he would play the game. Thus he realises that he would never choice I_1 at every round or I_2 at every round. He would play a mixture of these strategies - sometimes I_1, sometimes I_2. The strategy which is described by a certain probability to choose which strategy to use in the round is called mixed strategy. Thus if player I use p_1 as a probability to choose I_1 and p_2 to choose I_2, then his expected reward is given by $p_1a_{11} + P_2a_{21}$ against I_1 for the player II and $p_1a_{12} + P_2a_{22}$ against I_2 for player II.

A strategy that does not involve probabilities to choose the options is called pure strategy. For a game where the players has n pure strategies, the set of mixed strategy S_n can be represented by n-tiples $X = (x_1, x_2, \ldots, x_n)$, where $x_i \geq 0$, $i = 1,2,\ldots,n$ and $\sum_{i=1}^{n} x_i = 1$ i.e

$X \in S_n$; $S_n = \{P = (p_1, p_2, \ldots, p_n) ; \sum_{i=1}^{n} p_i = 1 , p_i \geq 0\}$. This corresponds to the mixed strategy where x_i is a probability of choosing I_i.

Let us consider a game with nxn payoff matrix A, the strategy X is said to be evolutionarily stable strategy (ESS) if and only if the following two conditions are satisfied

(i) equilibrium condition

$$Y . AX \leq X . AX \quad \text{for all } Y \in S_n \tag{1}$$

(ii) stability condition

$$\text{if } X \neq Y \text{ and } Y . AX = X . AX \text{, then } Y . AY < X . AY \tag{2}$$

where X. AY represents the payoff of a player using X strategy when matched against an opponent using strategy Y .

Condition (i) is the definition of a Nash equilibrium, a central notion in game theory. If X is a Nash equilibrium, then X is a best reply against itself (this property alone, however, does not guarantee non-invadability, since it permits that another strategy Y is an alternative best reply) . Condition (ii) (the stability condition) means that in this case X is better against Y than Y is against itself.

A strategy which satisfies condition (i) and (ii) is defined by Maynard Smith (1982) as a strategy which if an infinite homogeneous population adopts, then it can not be invaded by mutants under the action of natural selection

Let us assume that $Y = e_i$ where e_i is the vector that has one in the position i and 0 at the other position , then

$$Y \,.\, AX = (AX)_i = \sum_{j=1}^{n} a_{ij}\; x_j$$

In this case (1) becomes

$$(AX)_i \leq X \,.\, AX \qquad ; i = 1, 2, \ldots, n \tag{3}$$

which implies

$$p_i\,(AX\,)_i \;\leq p_i\, X \,.\, AX \qquad ;\; i = 1, 2, \ldots, n \;\;, p_i = x_i \tag{4}$$

By taking the summation in (4) , we get that

$$\sum_{i=1}^{n} p_i\,(AX)_i \leq X \,.\, AX \sum_{i=1}^{n} p_i$$

i.e $X . AX \leq X . AX$

Hence, the equality of (3) must hold for all i with $p_i > 0$.The Nash equilibrium condition (i) means that there exists a constant c such that $(AX)_i \leq c$ for all i , with equality if $p_i > 0$.

Therefore, the strategy $X \in \text{int } S_n$ is a Nash equilibrium iff its coordinates x_i satisfy

$$(AX)_1 = (AX)_2 = \cdots = (AX)_n$$

$$x_1 + x_2 + \cdots + x_n = 1$$

Game dynamics :

Let us assume that the population consists of n different types E_i to E_n , and x_i (t) is the frequency of type E_i at time t, so that the state of the population at time t is given by a vector X(t), where X(t) has non negative components summing up to one, and therefore belongs to the simplex S_n . If A is the nxn-matrix whose element a_{ij} is the average increase in fitness for an individual of type E_i encountering an individual of type E_j , and if individual meets randomly , then the average increase in fitness for an individual type E_i within the population is

$$\sum_{i=1}^{n} a_{ij} \ x_j = (AX)_i$$

and the average increase of fitness within the population is

$$\sum_{i=1}^{n} x_i \, (AX)_i \ = X . AX$$

We interpret each type as a strategy and its payoff as increase in fitness ; i.e in reproductive success .

If the population is very large, and if the generation blend continuously into each other, we may assume that the state X(t) evolves on S_n as a differentiable function of t, using the usual game dynamic of Taylor and Jonker , see Hofbauer and Sigmund [13] .

The rate of increase $\frac{\dot{x}_i}{x_i}$ of the type E_i is a measure of its evolutionary success given by the difference between the fitness $(AX)_i$ of E_i and the average fitness of the population . Thus we obtain

$$\frac{\dot{x}_i}{x_i} = (AX)_i - X \, . \, AX$$

which yields the game dynamical equation

$$\dot{x}_i = x_i \, [(AX)_i - X \, . \, AX] \tag{5}$$

This equation describes the action of selection upon the frequencies of the competing strategies .

Hence the equilibria of (5) in int S_n are the solutions of

$$(AX)_1 = (AX)_2 = \cdots = (AX)_n ,$$
$$x_1 + x_2 + \cdots + x_n = 1.$$

We note in particular that if some strategies are missing in the population, they are not introduced at a later time. Thus the models which we use are closed in the sense that they do not admit the emergence of missing types. In particular, the corners of the state simplex , i.e the vectors e_i of the standard basis, are equilibrium points. They correspond to pure states consisting of type E_i only.

Chapter I

The evolution of stochastic strategies in the Prisoner's Dilemma game

1.1 Introduction:

The theory of evolution is based on the struggle for survival.The Prisoner's Dilemma game (PD) is an example for the evolution of cooperation based on reciprocity (see [2],[4]). The Prisoner's Dilemma game is a non-zero-sum game which involves two individuals. Each of them can choose either to cooperate C or to defect D in discrete interactions. Consider for example, the case of two persons accused of a joint crime. In this case each of them has two decisions, either to confess to the police or not, where he dont know what is the other person's decision. If he keeps silent, this means he cooperates with his colleague. If he confesses (cooperates with the police), this means that he defects

.

We assume a symmetric payoffs of the Prisoner's Dilemma game. If both players cooperate, each gains R (for Reward), while if both players defect, each gain P (for Punishment), which is smaller than R. If one cooperates and the other defect, then the defector will get the highest payoff T (Temptation to defect), while the cooperator will get the lowest payoff S (Sucker's). It is clear that for one round, defection is the only best choice. The defector-player will receive T if the other player cooperates, while the cooperator-player will receive R if the other player cooperates, where R is less than T. And the defector-player will receive P if the other player defects, while the cooperator-player will receive S if the other player defects, where S is less than P.

The Dilemma will be produced if the two players chose defection. In this case each player will obtain P which is less than the R that they will get if they cooperate.

There is a possible varity which will occur if the game is iterated (Iterared Prisoner's Dilemma, IPD).To explain this, let us consider that w is the probability to repeat the game. If $w = 1$ or is sufficiently large the other player is using a strategy which consists in always defect, then the best reply is also to always defect. But if the strategy of the other player