STATISTICAL TREATMENT OF SOME QUEUEING PROBLEMS

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2 hours weekly for two semesters.

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2 hours weekly for two semesters.

CONTENTS

PREFACE

This thesis deals with Statistical Treatment of Some Queueing Problems. It consists of four chapters.

In the first chapter, we have studied the equilibrium behaviour of the G/M/m/s queue with finite waiting places. The asymptotic distribution of the number of customers at the moment of an arrival and at an arbitrary moment has been obtained. Also, we have obtain the waiting time and virtual waiting time distributions. Finally some special cases have been studied.

In the second chapter, we have obtained the closed form solutions of the joint equilibrium distribution of queue size for large classes of M/G/1/N with certain type of service loop under the queueing discipline that is first-come, first-served. The classes of G/M/k/N queues are obtained as special cases of load-dependent servers, and we note that the solution of the system depends only on the distribution of the general server and not on the number of customers in the system. Finally some special cases also have been obtained.

The third chapter deals with the single server queueing system with finite waiting room. The phase technique is used to obtain the distribution function of the queue length at instants at which customers complete service, under the assumption that a customer who arrives to find the waiting room full departs without waiting for service. The limit as the size of the waiting room becomes infinite is found.

In the fourth chapter we deal with the maximum likelihood point estimation for the interarrival rate, the service rate and the utilization factor of (i) the m-server loss queue: M/M/m/m, (ii) the self-regulating queue with finite customer population M/M/1/M, (iii) the finite customer population queue with infinite number of servers : $M/M/\infty/M$, (iv) the Erlangian queue: $M/E_r/1$, (v) the Erlangian queue: $E_r/M/1$, and at last (vi) the bulk arrival queue.

CHAPTER I

MATHEMATICAL TREATMENT OF THE G/M/m/Δ QUEUE WITH FINITE WAITING PLACES

This chapter studies the equilibrium behaviour of the G/M/m queue with only Δ waiting places. We start by studying the joint distribution of the number of customers present at time t and the time elapsing until the next arrival after t. This gives the asymptotic distribution of the number of customers at the moment of an arrival and at an arbitrary moment. The waiting time and virtual waiting time distributions are easily obtained. Finally some special cases also have been obtained.

1.1. INTRODUCTION AND FORMULATION OF THE PROBLEM

Queues with finite waiting places have obtained relatively little attention in the literature. Keilson [23] derived several results on the single server queueing systems M/G/1 and G/M/1. The M/G/1 queue is also studied by Cohen [4]. Takács [51] treated the many servers queueing system G/M/m, in the present chapter we study the same model. The results of Takács are given in another form, and are also extended.

We now assume that the intervals of time elapsing between two arrivals are independent random variables with p.d.f. $a(x)$. Further define

$$a_1 = \int_0^\infty x\, a(x)\, dx < \infty, \tag{1.1.1}$$

$$A^*(\theta) = \int_0^\infty e^{-\theta x}\, a(x)\, dx. \tag{1.1.2}$$

At the service point there are m servers, and the service times are exponentially distributed with parameter β. The number of waiting places equals Δ. If at the moment of an arrival the number of customers is $m + \Delta$, the arriving customer is not admitted to the system. He disappears and never returns (is lost).

This system is treated by a supplementary variable technique, which we earlier have used to study the M/G/1 queue. Thus in the study of queue length, we use the time until the next arrival as supplementary variable.

The asymptotic distribution of the number of customers present is derived (both at an arbitrary moment and at the moment of an arrival). Further, the mean waiting time and the mean number of customers is found. We also consider the special case m=1 and the transient behaviour is derived. Finally we comment on the G/M/m queue with infinite waiting places.

1.2. THE CONNECTION BETWEEN THE TIME-CONTINUOUS AND THE IMBEDDED PROCESS

Let N(t) denote the number of customers present at time t, U(t) equal the period of time which we must wait until the next arrival after time t. Observe that we register every arriving customer, also those who are not admitted to the system. The state of the system at time t is now defined by {N(t), U(t)}. Their joint distribution is given by

$$p_k(u,\ t)\ du = P\{(N(t)=k) \cap (u < U(t) \le u+du)\}$$

$$u \ge 0,\ \ k=0,1,\ldots,m+\Delta. \qquad (1.2.1)$$

Relating the state of the system at time t and t+dt, we obtain by an ordinary argument that under certain regularity assumptions

$$p_0(u,t+\Delta) = p_0(u+\Delta,t) + \beta\,\Delta\, p_1(u+\Delta,t),$$

$$p_k(u,t+\Delta) = (1-k\beta\Delta)\, p_k(u+\Delta,t) + \alpha(u)\,\Delta\, p_{k-1}(0,t) + (k+1)\,\beta\,\Delta\, p_{k+1}(u+\Delta,t),$$

$$p_k(u,t+\Delta) = (1-m\beta\Delta)\, p_k(u+\Delta,t) + \alpha(u)\,\Delta\, p_{k-1}(0,t) + m\,\beta\,\Delta\, p_{k+1}(u+\Delta,t),$$

$$p_{m+\vartriangle}(u,t+\Delta) = (1-m\beta\Delta)\, p_{m+\vartriangle}(u+\Delta,t) + \alpha(u)\,\Delta\, p_{m+\vartriangle-1}(0,t) + \alpha(u)\,\Delta\, p_{m+\vartriangle}(0,t),$$

dividing by Δ and taking the limit as $\Delta \longrightarrow 0$ we have

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right) p_0(u,t) = \beta\, p_1(u,t), \qquad (1.2.2)$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right) p_k(u,t) = -\,k\,\beta\, p_k(u,t) + (k+1)\,\beta\, p_{k+1}(u,t) + \alpha(u)\, p_{k-1}(0,t), \quad k=1,2,\ldots,m-1, \qquad (1.2.3)$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right) p_k(u,t) = -\,m\,\beta\, p_k(u,t) + m\,\beta\, p_{k+1}(u,t) + \alpha(u)\, p_{k-1}(0,t), \quad k=m,m+1,\ldots,m+\vartriangle-1, \qquad (1.2.4)$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right) p_{m+\vartriangle}(u,t) = -\,m\,\beta\, p_{m+\vartriangle}(u,t) + \alpha(u)\left(p_{m+\vartriangle-1}(0,t)+p_{m+\vartriangle}(0,t)\right). \qquad (1.2.5)$$

As we will restrict ourselves now to study the asymptotic distribution, we let $t\longrightarrow\infty$, and thus derivatives with respect to t tend to zero in (1.2.2)-(1.2.5).

Let

$$p_k(u) = \lim_{t\longrightarrow\infty} p_k(u,t), \quad k=0,1,\ldots,m+\vartriangle, \qquad (1.2.6)$$

which gives the asymptotic distribution of $\{N(t), U(t)\}$.

Also define

$$P_k^*(\theta) = \int_0^\infty p_k(u)\, e^{-\theta u}\, du, \qquad k=0,1,\ldots,m+\delta\ . \tag{1.2.7}$$

From (1.2.2) by taking the limit as $t \longrightarrow \infty$ we have

$$-\frac{\partial}{\partial u} p_0(u) = \beta\, p_1(u)$$

multiplying both side by $e^{-\theta u}$ and integrating with respect to u from zero to ∞ we have

$$-\int_0^\infty e^{-\theta u} \left(\frac{\partial}{\partial u} p_0(u)\right) du = \beta \int_0^\infty p_1(u)\, e^{-\theta u}\, du$$

$$-\{[e^{-\theta u}\, p_0(u)]_0^\infty + \theta \int_0^\infty p_0(u)\, e^{-\theta u}\, du\} = \beta \int_0^\infty p_1(u)\, e^{-\theta u}\, du.$$

Using (1.2.7) we get

$$p_0(0) - \theta\, P_0^*(\theta) = \beta\, P_1^*(\theta)$$

thus,

$$\theta\, P_0^*(\theta) = p_0(0) - \beta\, P_1^*(\theta)\ , \tag{1.2.8}$$

from (1.2.3) by taking the limit for both side as $t \longrightarrow \infty$ and using (1.2.6) we get

$$-\frac{\partial}{\partial u} p_k(u) = -\,k\,\beta\, p_k(u) + (k+1)\,\beta\, p_{k+1}(u) + \alpha(u)\, p_{k-1}(0)$$

multiplying both side by $e^{-\theta u}$ and integrating with respect to u from zero to ∞ we have

$$-\int_0^\infty e^{-\theta u} \left(\frac{\partial}{\partial u} p_k(u)\right) du = -\,k\,\beta \int_0^\infty p_k(u)\, e^{-\theta u}\, du$$

$$+\ (k+1)\,\beta \int_0^\infty p_{k+1}(u)\, e^{-\theta u}\, du + p_{k-1}(0) \int_0^\infty \alpha(u)\, e^{-\theta u}\, du$$

using (1.2.7) and (1.1.2) we get

$$P_k(0) - \theta\, P_k^*(\theta) = -\,k\,\beta\, P_k^*(\theta) + (k+1)\,\beta\, P_{k+1}^*(\theta) + A^*(\theta)\, p_{k-1}(\theta)$$

$$(\theta-k\beta)\ P_k^*(\theta) = p_k(0) - A^*(\theta)\ p_{k-1}(0) - (k+1)\ \beta\ P_{k+1}^*(\theta),$$
$$k=1,2,\ldots,m-1\ , \qquad (1.2.9)$$

from (1.2.4) by taking the limit for both side as $t \longrightarrow \infty$ and using (1.2.6) we have

$$-\frac{\partial}{\partial u}\ p_k(u) = -\ m\ \beta\ p_k(u) + m\ \beta\ p_{k+1}(u) + a(u)\ p_{k-1}(0)$$

multiplying both side by $e^{-\theta u}$ and integrating with respect to u from zero to ∞ we have

$$-\int_0^\infty e^{-\theta u}\ (\frac{\partial}{\partial u}\ p_k(u))\ du = -\ m\ \beta \int_0^\infty p_k(u)\ e^{-\theta u}\ du$$

$$+\ m\ \beta \int_0^\infty p_{k+1}(u)\ e^{-\theta u}\ du + p_{k-1}(0) \int_0^\infty a(u)\ e^{-\theta u}\ du.$$

Using (1.2.7) and (1.1.2) we get

$$p_k(\theta) - \theta\ P_k^*(\theta) = -\ m\ \beta\ P_k^*(\theta) + m\ \beta\ P_{k+1}^*(\theta) + A^*(\theta)\ p_{k-1}(0)$$

$$(\theta-m\beta)\ P_k^*(\theta) = p_k(0) - A^*(\theta)\ p_{k-1}(0) - m\ \beta\ P_{k+1}^*(\theta),$$
$$k=m,m+1,\ldots,m+\Delta-1, \qquad (1.2.10)$$

from (1.2.5) by taking the limit for both side as $t \longrightarrow \infty$ and using (1.2.6) we have

$$-\frac{\partial}{\partial u}\ p_{m+\Delta}(u) = -\ m\ \beta\ p_{m+\Delta}(u) + a(u)\ p_{m+\Delta-1}(0) + a(u)\ p_{m+\Delta}(0),$$

multiplying both side by $e^{-\theta u}$ and integrating with respect to u from zero to ∞ we have

$$-\int_0^\infty e^{-\theta u}\ (\frac{\partial}{\partial u}\ p_{m+\Delta}(u))\ du = -\ m\ \beta \int_0^\infty p_{m+\Delta}(u)\ e^{-\theta u}\ du$$

$$+\ p_{m+\Delta-1}(0) \int_0^\infty a(u)\ e^{-\theta u}\ du + p_{m+\Delta}(0) \int_0^\infty a(u)\ e^{-\theta u}\ du,$$

Using (1.2.7) and (1.1.2) we get

$$p_{m+\Delta}(0) - \theta\ P_{m+\Delta}^*(\theta) = -\ m\ \beta\ P_{m+\Delta}^*(\theta) + A^*(\theta)\ p_{m+\Delta-1}(0) + A^*(\theta)\ p_{m+\Delta}(0)$$

Thus

$$(\theta - m\beta)\, P^*_{m+\Delta}(\theta) = p_{m+\Delta}(0) - A^*(\theta)(p_{m+\Delta-1}(0) + p_{m+\Delta}(0)) . \qquad (1.2.11)$$

Since

$$q_k = p_k(0) \left(\sum_{j=0}^{m+\Delta} p_j(0) \right)^{-1} ,$$

is the probability of k customers present immediately before an arrival. In order to obtain $\sum_{j=0}^{m+\Delta} p_j(0)$, we sum all equations (1.2.8) - (1.2.11).

From (1.2.8)

as k=0; $\theta P^*_0(\theta) = p_0(0) - \beta P^*_1(\theta)$.

From (1.2.9) we have this system of equations

as k=1; $\theta P^*_1(\theta) = p_1(0) - A^*(\theta)\, p_0(0) + \beta P^*_1(\theta) - 2\beta P^*_2(\theta)$,

as k=2; $\theta P^*_2(\theta) = p_2(0) - A^*(\theta)\, p_1(0) + 2\beta P^*_2(\theta) - 3\beta P^*_2(\theta)$,

... ...

as k=m-2; $\theta P^*_{m-2}(\theta) = p_{m-2}(0) - A^*(\theta) p_{m-3}(0) + (m-2)\beta P^*_{m-2}(\theta) - (m-1)\beta P^*_{m-1}(\theta)$;

as k=m-1; $\theta P^*_{m-1}(\theta) = p_{m-1}(0) - A^*(\theta) p_{m-2}(0) + (m-1)\beta P^*_{m-1}(\theta) - m\beta P^*_m(\theta)$.

From (1.2.10) we have this systems of equations

as k=m; $\theta P^*_m(\theta) = p_m(0) - A^*(\theta) p_{m-1}(0) + m\beta P^*_m(\theta) - m\beta P^*_{m+1}(\theta)$,

as k=m+1; $\theta P^*_{m+1}(\theta) = p_{m+1}(0) - A^*(\theta) p_m(0) + m\beta P^*_{m+1}(\theta) - m\beta P^*_{m+2}(\theta)$,

... ...

as k=m+Δ-2; $\theta P^*_{m+\Delta-2}(\theta) = p_{m+\Delta-2}(0) - A^*(\theta)\, p_{m+\Delta-3}(0) + m\beta P^*_{m+\Delta-2}(\theta) - m\beta P^*_{m+\Delta-1}(\theta)$,

as k=m+Δ-1; $\theta P^*_{m+\Delta-1}(\theta) = p_{m+\Delta-1}(0) - A^*(\theta)\, p_{m+\Delta-2}(0) + m\beta P^*_{m+\Delta-1}(\theta) - m\beta P^*_{m+\Delta}(\theta)$.

From (1.2.11) we have

as $k=m+\delta$; $\theta\, P^{*}_{m+\delta}(\theta)=p_{m+\delta}(0) - A^{*}(\theta)\, p_{m+\delta-1}(0) - A^{*}(\theta)\, p_{m+\delta}(\theta)$.

Summing he above system of equations we get,

$$\theta \sum_{k=0}^{m+\delta} P^{*}_{k}(\theta) = \sum_{k=0}^{m+\delta} p_{k}(0) - A^{*}(\theta) \sum_{k=0}^{m+\delta} p_{k}(0) \quad ,$$

$$\sum_{k=0}^{m+\delta} P^{*}_{k}(\theta) = \frac{1-A^{*}(\theta)}{\theta} \sum_{k=0}^{m+\delta} p_{k}(0) \quad . \tag{1.2.12}$$

Since $P^{*}_{k}(0) = \int_{0}^{\infty} p_{k}((u)du=p_{k}$, is the probability that there exist k customers in the queue and $\sum_{k} p_{k} = 1$, then $\sum_{k=0}^{m+\delta} P^{*}_{k}(0) = 1$, we must have

$$\sum_{k=0}^{m+\delta} p_{k}(0) = \frac{1}{a_{1}} \quad . \tag{1.2.13}$$

Thus the distribution of the imbedded process (immediately before an arrival) is given by

$$q_{k} = p_{k}(0)\, (\frac{1}{a_{1}})^{-1}$$

$$= a_{1}\, p_{k}(0) \quad , \quad k=0,1,\dots,m+\delta \quad . \tag{1.2.14}$$

The probability of k customers at an arbitrary moment is given by

$$p_{k} = P^{*}_{k}(0) \; , \qquad k=0,1,\dots,m+\delta \quad . \tag{1.2.15}$$

Now to determine the connection between these two distributions, we insert $\theta=0$ in(1.2.8)-(1.2.10). Then we have

$$p_{0}(0) - \beta\, p_{1} = 0 \quad , \tag{1.2.16}$$

$$p_{k-1}(0) - k\, \beta\, p_{k} = p_{k}(0) - (k+1)\, \beta\, p_{k+1} \quad , \qquad k=1,2,\dots,m-1 \; , \tag{1.2.17}$$

$$p_{k-1}(0) - m\, \beta\, p_{k} = p_{k}(0) - m\, \beta\, p_{k+1} \quad , \qquad k=m,m+1,\dots,m+\delta-1 . \tag{1.2.18}$$