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On Evolutionary Dynamics of Repeated Games

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(Pure Mathematics)

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Summary

This work deals with a branch of game theory, namely evolutionary game dynamics. Which has been created by Maynard Smith, J., Zeeman, E. C. and others.

The aim of this work is to study the Prisoner's Dilemma game (PDG), especially the randomly alternating (RA) model of this game. This game has two players and two choices, and therefore we have four possible outcomes, these outcomes lead to 16 strategies (2^4). We have calculated the adaptive dynamics for this model. Also, the 16×16 payoff matrix is computed in case of occurring a small error probability ϵ for the same game.

This thesis consists of three chapters. In the first chapter an introduction was given about game theory. It concerns about games in general, their types and their analytical methods. We mention some important definitions, theorems and famous examples in game theory.

In the second chapter we introduce the simultaneous and alternating models of PD game. The transition matrix of RAPDG was determined. Also, the payoff for a player in RAPDG was computed. In this chapter we have a homogeneous population of strategy Ω. If an individual was permitted a small deviation from strategy Ω, which direction would be most favorable. Any parameter s in Ω changes according to the adaptive dynamics $\dot{s} = \frac{\partial F}{\partial s}$, where F is the payoff for Ω-player. We deduce the adaptive dynamics for the RAPDG, and apply results to some of most famous strategies for this game. The results of this chapter were published in [11].

In the third chapter, we introduce the transition rule of each automaton of a two state automata repeated game, which depends on the initial state of the game and as well on the outcome of previous round. We explain the method of computing the payoff matrix for each different initial state of the automata. We mention the definitions of Markov matrix (transition matrix) and the stationary probability distribution of Markov matrix. The 16×16 payoff matrix is computed in case of occurring an error in implementation and due to the error in perception for RAPDG. Then we studied all the strategies to know the best replies, and the results are published in [12].

History of Game Theory

History of Game Theory

The first important text in game theory is the "*Theory of Games and Economic Behavior*" (see [25]). Game theory has evolved considerably since the publication of this book and its reach has extended far beyond the confines of mathematics. This is due in a large part to contributions in the 1950s from John Nash (1950, 1951). However, it was in the 1970s that game theory as a way of analysing strategic situations began to be applied in all sorts of diverse areas including economics, politics, international relations, business and biology. A number of important publications precipitated this breakthrough, however, and **Thomas Schelling's** book *The Strategy of Conflict* (1960) still stands out from a social science perspective. **Hutton** (1996) described game theory as 'an intellectual framework for examining what various parties to a decision should do given their possession of inadequate information and different objectives'. This definition describes what game theory can be used for rather than what it is. It also implicitly characterizes the distinctive features of a situation that make it amenable to analysis. These features are that the actions of the parties concerned impact on each other but exactly how this might happen is unknown. Interdependence and information are therefore critical aspects of the definition of game theory (see [8]).

Chapter 1 : Concepts and Theorems in Game Theory

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1.1 What is game theory?

Game theory is a technique used to analyse situations where for two or more individuals (or institutions) the outcome of an action by one of them depends not only on the particular action taken by that individual but also on the actions taken by the other (or others). In these circumstances the plans or strategies of the individuals concerned will be dependent on expectations about what the others are doing. Thus individuals in these kinds of situations are not making decisions in isolation; instead they make decisions interdependently. This is called *strategic interdependence* and such situations are commonly known as *games of strategy*, or simply *games*, while the participants in such games are referred to as *players*. In strategic games the actions of one individual or group impact on others and, crucially, the individuals involved are aware of this (see [8]).

Because players in a game are conscious that the outcomes of their actions are affected by and affect others they need to take into account the possible actions of these other individuals when they themselves make decisions. However, when individuals have limited information about other individuals' planned actions (their *strategies*); they have to make conjectures about what the opponent will do. These kinds of thought processes constitute strategic thinking and when this kind of thinking is involved game theory can help us to understand what is going on and make predictions about likely outcomes (see [8]).

1.2 Describing Strategic Games

In order to be able to apply game theory the first step is to define the boundaries of the strategic game under consideration. Games are defined in terms of their rules. The rules of the game incorporate information about the players' identity and their knowledge of the game, their possible moves or actions and their *payoffs*. The rules of a game describe in detail how one player's behavior impacts on other players' payoffs. A player can be an individual, a couple, a family, a firm, a pressure group, a government, an intelligent animal in fact any kind of thinking entity that is generally assumed to act rationally (rationality implies that every player is motivated by maximizing his own payoff) and is involved in a strategic game with one or more other players (see [8]).

Players' payoffs may be measured in terms of units of money or time, chocolate or anything that might be relevant to the situation. However, it is often useful to generalize the representation of payoff in terms of units of satisfaction or utility. *Utility* is an abstract, subjective concept and its use is widespread in economics (see [8]).

My utility from, say, a bar of chocolate is likely to be different from yours and anyway the two will not be directly comparable, but if we both prefer chocolate to pizza we will both derive more utility from the former. When a strategic situation is modelled as a game and the payoffs are measured in terms of units of utility (sometimes called utils) then these will need to be assigned to the payoffs in a way that makes sense from the player's perspectives. What usually matters most is the ranking between different alternatives. Thus if a bar of chocolate makes you happier than a pizza the number of utility units assigned to the former should be higher. The actual number of units assigned will not always be important. Sometimes it is simpler not to assign numbers to payoffs at all. Instead we can assign letters or symbols to payoffs and then stipulate their rankings. For example, instead of assigning a payoff of, say, ten to a bar of chocolate and three to a pizza, we could simply assign the letter A to the chocolate and the letter B to the pizza and specify that A is greater than B (i.e. $A > B$). This can be quite a useful simplification when we want to make general observations about the structure of a game. However, in some circumstances the actual value of the payoffs is important and then we need to be a bit more precise (see [8]).

Rational individuals are assumed to prefer more utility to less and therefore in a strategic game a payoff that represents more utility will be preferred to one that represents less. Note that while this will always be true about levels of satisfaction or pleasure it will not always be the case when we are talking about quantities of material goods like chocolate it is possible to eat too much chocolate. Players in a game are assumed to act rationally if they make plans or choose actions with the aim of securing their highest possible payoff (i.e. they choose strategies to maximize payoffs). This implies that they are self-interested and pursue aims. However, because of the interdependence that characterizes strategic games, a player's best plan of action for the game, their preferred strategy, will depend on what they think the other players are likely to do (see [8]).

The theoretical outcome of a game is expressed in terms of the strategy combinations that are most likely to achieve the players' goals given the information available to them. Game theorists focus on combinations of the players' strategies that can be characterized as *equilibrium* strategies. If the players choose their equilibrium strategies they are doing the best they can give the other players' choices. In these circumstances there is no incentive for any player to change their plan of action. The equilibrium of a game describes the strategies that rational players are predicted to choose when they interact. Predicting the strategies that the players in a game are likely to choose implies we are also predicting their payoffs (see [8]).

Games are often characterized by the way or order in which the players move. Games in which players move at the same time or their moves are hidden are called *simultaneous-move* or *static* games. Games in which the players move in some kind of predetermined order are call *sequential move* game or *dynamic* game. These two types of games are discussed in the following sections (see [8]).

Definition 1.2.1: An *action* or *move* by player i, denoted a_i , is a choice he can make, and player i's action set, $A_i = \{a_i\}$, is the entire set of actions available to him, while an ordered set $A = \{a_i\}$, $(i = 1, \dots, n)$ is a set of one action for each of the n-players in the game (see [31]).

Definition 1.2.2: *Information set* is a concept which will be defined more precisely later. For now, think of a player's information set as his knowledge at a particular time of the values of different variables. The elements of the information set are the different values that the player thinks are possible. If the information set has many elements, there are many values the player cannot rule out; if it has one element, he knows the value precisely. A player's information set includes not only distinctions between the values of variables, but also knowledge of what actions have previously been taken, so his information set changes over the course of the game (see [31]).

1.3 Strategy

Player i's strategy s_i is a rule that tells him which action to choose at each instant of the game, given his information set. Player i's strategy set or strategy space $S_i = \{s_i\}$ is the set of strategies available to him. A strategy profile $s = (s_1, \dots, s_n)$ is an ordered set consisting of one strategy for each of the n-players in the game.

Since the information set includes whatever the player knows about the previous actions of other players, the strategy tells him how to react to their actions (see [31]).

1.3.1 Pure and Mixed Strategies

The simplest kind of strategy selects unambiguously some specific course of action (also referred to as a ‘move’); for example, ‘help an old person cross the road’, or ‘shoot an opponent’. This is called *a pure strategy*. However, there are times when you are uncertain about what is the best pure strategy. In these cases, you may choose as if at random between two or more pure strategies: for example, in the absence of reliable meteorological information, you may decide on whether to carry an umbrella by tossing a coin. This type of strategy is called *a mixed strategy*, in the sense that you choose a specific ‘probabilistic mix’ of a set of pure strategies (see [17]).

Definition 1.3.1: If a player has N available pure strategies $(s_1, s_2, \dots, s_N)$, a *mixed strategy* M is defined by the probabilities $(p_1, p_2, \dots, p_N)$ with which each of her pure strategies will be selected. Note that:

$$0 \le p_i \le 1, \sum_i p_i = 1, \text{ for } 1 \le i \le N$$

Note also that to choose a mixed strategy $(p_1 = 0, p_2 = 0, \ldots, p_j = 1, \ldots, p_N = 0)$ is equivalent to choosing pure strategy s_j (see [17]).

Definition 1.3.2: By player i's *payoff* $u_i(s_1, \ldots, s_n)$, we mean either:

(1) The utility player i receives after all players have picked their strategies and the game has been played out; or
(2) The expected utility he receives as a function of the strategies chosen by him and the other players (see [31]).

Example 1.3.1: ***The hawk-dove game.***

When animals engage in a conflict (over mates, land, food etc.) they may pick one of two strategies. Either they behave as hawks in this case they fight until one of them gets injured or the opponent flees. Or they behave as doves in this case they may display aggressive behavior, but they retreat as soon as their opponent shows signs of escalating the conflict into a fight. Assume that the winner of the contest gains $G > 0$ (a mate, some land, some food), and injury leads to a loss of $I > 0$ (no mate, no food, scratches, a decreased level of self-confidence etc.). We assume that the cost of an injury is larger than the value of the gain:

$$G - I < 0.$$

Let us consider the outcome of the 4 possible conflict situations:

1. When two hawks meet, there will be a fight with one winner and one loser, and the expected payoff for each hawk is $(G - I)/2$.
2. When a hawk meets a dove, the dove bails out, and the hawk receives $G > 0$.
3. When a dove meets a hawk, the dove flees, and gets nothing, but also does not get harmed, so his payoff is 0.
4. When a dove meets a dove, he might or might not flee, and his expected gain will be $G/2$.

We summarize these outcomes in the following table