Faculty of Education
Mathematics Department

On Digital Topology and its Applications

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(Pure Mathematics)

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This thesis contains three chapters:

- The introductory Chapter I contains the basic concepts and notations that are needed such as the topological space, supra-topology, semi-open set, λ-open set, median filter and its root image.
- In Chapter II, we give a summary about the digital topology and how can deduce it. Also, we show some properties of the digital topology such as the separation axioms, dense set, point-wise dense set, cut point, extremely disconnected, open and closed subsets.
- The aim of Chapter III is to find a relation between the digital topology and the root images of median filters. We use the concepts of semi-open and λ-open sets in Marcus-Wyse and Khalimsky topologies to introduce a class finer than the standard semi-topology and deduced a relation between these types of open sets and root images of median filters.

Contents

Chapter I

Preliminaries

The introductory chapter is considered as a background for the basic concepts and notations used in this thesis. The aim of this chapter is to give a short survey of some needed definitions and theories of the material used in the thesis. The basic concepts of the general topology, digital topology, and root images of median filters are investigated.

1.1. General topological concepts

In this section we will state some needed definitions and theories from the general topology (for more details see [9], [16], and [21]):

Definition 1.1.1 [9] The ***Cartesian product*** $X \times Y$ of the sets X and Y is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$.

Definition 1.1.2 [16] Any subset of the Cartesian product $X \times Y$ is called a ***relation*** and $(x, y) \in R$ means x is related to y in R (or by R).

Definition 1.1.3 [16], [21] A (binary) relation R in X i.e., from X to X is said to be:

1) ***reflexive*** if $(x, x) \in R$ for all $x \in X$.
2) ***symmetric*** if $(x, y) \in R$, then $(y, x) \in R$.
3) ***transitive*** if $(x, y) \in R$ and $(y, z) \in R$ together implies $(x, z) \in R$.
4) ***anti-symmetric*** if $(x, y) \in R$ and $(y, x) \in R$ together implies $x = y$.

5) ***equivalence*** if R is reflexive, symmetric, and transitive.

Definition 1.1.4 [16] If R is an equivalence relation in X, then the ***equivalence class*** of any element $x \in X$, denoted by $[x]$, is the set $[x] \coloneqq \{y \mid (x, y) \in R\}$

Definition 1.1.5 [16] The collection of equivalence classes, denoted by X/R, is called the ***quotient*** of X by R and given by: $X/R \coloneqq \{[x] \mid x \in X\}$.

Definition 1.1.6 [16] A class $\mathcal{A}$ of non-empty subsets of X is called a ***partition*** of X if

(1) Each $x \in X$ belongs to some member of $\mathcal{A}$, and

(2) The members of $\mathcal{A}$ are pair-wise disjoint.

Theorem 1.1.1 [16] Let R be an equivalence relation in X. Then the quotient set X/R forms a partition of X.

Definition 1.1.7 [9] A relation $f \subseteq X \times Y$ is called a ***function*** or a ***mapping*** from X to Y if for every $x \in X$ there exists a unique element $y \in Y$ such that $(x, y) \in f$.

Definition 1.1.8 [9], [21] A function f from X to Y is called:

1) ***one-to-one*** (injective) if for any $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.
2) ***onto*** (surjective) if $f(X) = Y$.
3) ***Bijective*** if it is both one-to-one and onto.

Definition 1.1.9 [21] The rule $f^{-1}(y) = x \iff f(x) = y$ defines a function $f^{-1}: Y \to X$ iff f is bijective. This function is called the ***inverse function*** of f.

Definition1.1.10 [9] A ***topological space*** is a pair (X, τ) consisting of a non-empty set X and a family τ of subsets of X satisfying the following conditions:

(1) $\phi \in \tau$ and so do X.

(2) If $A \in \tau$ and $B \in \tau$, then $A \cap B \in \tau$.

(3) If $\mathcal{A} \subseteq \tau$, then $\cup \mathcal{A} \in \tau$.

The set X is called a space, the elements of X are called points of the space, and the subsets of X belonging to τ are called open in the space; the family τ of open subsets of X is also called a topology on X. A point $x \in X$ is called open point if $\{x\}$ is an open set. The set of all open points of a subset A of X is denoted by A_{op}.

Definition 1.1.11 [3], [6], [11] A topological space (X, τ) is said to be ***Alexandroff topology*** if τ satisfies the following additional axiom:

(2^*) The arbitrary intersection of open sets is open.

According to this additional axiom, every point possesses a smallest open set. Since the open and the closed sets in an *Alexandroff* topology satisfy the same axioms, then there is a complete symmetry. Instead of calling the open sets open, we may call them closed, and call the closed sets open. Then we get the ***dual*** of the *Alexandroff* topology [10].

Definition 1.1.12 [21] Let (X, τ) be a topological space and $A \subseteq X$. Then $\tau_A = \{G \cap A \mid G \in \tau\}$ is a topology on A, called a topology induced by A or the ***relative topology*** on A and (A, τ_A) is called ***subspace*** of (X, τ).

Definition 1.1.13 [21] Let (X, τ) be a topological space and $A \subseteq X$. The set $int(A) = \cup \{G \in \tau \mid G \subseteq A\}$ is called the ***interior*** of A.

Proposition 1.1.1 [9] Let (X, τ) be a topological space and $A \subseteq X$. Then $x \in int(A)$ iff there is O_x such that $O_x \subseteq A$.

Definition 1.1.14 [21] A subset A of a topological space (X, τ) is said to be a ***neighborhood*** of p if there is O_p such that $O_p \subseteq A$.

Definition 1.1.15 [21] Let (X, τ) be a topological space and $A \subseteq X$. The set $Cl(A) = \cap \{F \mid F \supseteq A, F^c \in \tau\}$ is called the ***closure*** of A.

Proposition 1.1.2 [9] Let (X, τ) be a topological space and $A \subseteq X$. Then $x \in Cl(A)$ iff for all O_x, $O_x \cap A \neq \phi$.

Definition 1.1.16 [9] A subset A of a topological space (X, τ) is called ***dense*** in X if $Cl(A) = X$.

Definition 1.1.17 [27] A subset A a topological space (X, τ) is called ***point wise dense*** in X if $\cup \{Cl\{x\} | x \in A$ and $\{x\}$ is open$\} = X$.

Definition 1.1.18 [21] A topological space (X, τ) is called ***door space*** if every subset of X is either open or closed.

Definition 1.1.19 [1], [15], [23], [27] Let (X, τ) be a topological space. A subset A of X is called ***semi-open*** set if $A \subseteq Cl(int(A))$.

Definition 1.1.20 [1], [23], [27] A subset A of a topological space (X, τ) is called ***semi-closed*** if its complement A^c is semi-open set.

Definition 1.1.21 [1], [27] Let (X, τ) be a topological space. The ***semi-closure*** of a subset A of X, denoted by $Cl_s(A)$, is the intersection of all semi-closed supersets of A.

Definition 1.1.22 [1], [27] Let (X, τ) be a topological space. The ***semi-interior*** of a subset A of X, denoted by $int_s(A)$, is the union of all semi-open subsets of A.

Proposition 1.1.3 Let (X, τ) be a topological space and let $A \subseteq X$. Then

(1) $x \in int_s(A)$ if and only if there is O_x^s such that $O_x^s \subseteq A$.

(2) $x \in Cl_s(A)$ if and only if $O_x^s \cap A \neq \phi$ for all O_x^s.

Definition 1.1.23 A subset A of a topological space (X, τ) is called ***regular semi-open*** if $A = int_s(Cl_s(A))$.

Definition 1.1.24 [19] Let B be a subset of a topological space (X, τ). Then B is called ***Λ-set*** (respectively ***V-set***) if $B = B^{\Lambda}$ (respectively $B = B^{\mathrm{V}}$) where:

$$B^{\Lambda} = \cap \{G \mid G \supset B, G \in \tau\} \text{ and } B^{V} = \cup \{F \mid B \supset F, F^{c} \in \tau\}$$

Definition 1.1.25 [5] A subset A of a topological space (X, τ) is called ***λ-closed*** if $A = G \cap F$ where G is a Λ-set and F is a closed set.

Definition 1.1.26 [22] A subset A of a topological space (X, τ) is called ***λ-open*** if its complement is λ-closed.

Theorem 1.1.4 [22] Let (X, τ) be a topological space and $A \subseteq X$. The following statements are equivalent:

(1) A is λ-open.

(2) $A = H \cup G$, where H is a V-set and G is open set.

Note that, in *Alexandroff* topology, A subset A of X is called ***λ-open*** if $A = G \cup H$ where G is an open set and H is a closed set.

Proposition 1.1.4 [22] Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in Cl_{\lambda}(A)$, the ***λ-closure*** of A, if for every O_x^{λ} , $A \cap O_x^{\lambda} \neq \phi$.

Proposition 1.1.5 [22] Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in int_{\lambda}(A)$, the ***λ-interior*** of A, if there exists O_x^{λ} such that $O_x^{\lambda} \subseteq A$.

Definition 1.1.29 A subset A of a topological space (X, τ) is called ***regular λ-open*** if $A = int_{\lambda}(Cl_{\lambda}(A))$.

Definition 1.1.30 [9] A family $\beta \subseteq \tau$ is called a ***base*** for the topological space (X, τ) if every non-empty open subset of X can be represented as a union of a subfamily of β.

Theorem 1.1.5 [21] Let $\beta \subseteq P(X)$. Then β is a base for a topological space (X, τ) iff the following conditions are satisfied:

(1) $\cup \{B | B \in \beta\} = X$.

(2) Given $B_1, B_2 \in \beta$ and $x \in B_1 \cap B_2$, then there is $B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Definition 1.1.31 [9] A family $\eta \subseteq \tau$ is called a ***sub base*** for the topological space (X, τ) if the family of all finite intersections of members of η forms a base for (X, τ).

Definition 1.1.32 [9] A family $\beta(x)$ is called a ***base*** for the topological space (X, τ) ***at the point*** x (or ***local base***) if for any O_x there is $V \in \beta(x)$ such that $x \in V \subseteq O_x$.

Note that if for any $x \in X$, a base $\beta(x)$ for (X, τ) at the point x is given, then the union $\beta = \bigcup_{x \in X} \beta(x)$ is a base for (X, τ).

Definition 1.1.33 [6], [10] Let (X, τ) be a topological space. If there is a base β for the topology such that for any other base γ it holds that $\beta \subseteq \gamma$, then we say that β is the ***smallest base*** (or a ***unique minimal base***) for (X, τ).

It is clear that *Alexandroff* topology has a smallest base which is $\beta = \{N_{min}(x) \mid x \in X\}$ where $N_{min}(x)$ is the intersection of all open sets containing x. The existence of a smallest base is not sufficient for the space to be Alexandroff topology. For example, let $X = \mathbb{R}$ and $\tau = \left\{\left[0, \frac{1}{n}\right); n = 1,2, \dots\right\} \cup \{X, \emptyset\}$. It is easy to see that the topology itself is a smallest base, but the intersection of all open sets containing 0 is $\{0\}$ which is not an open set [10].

Definition 1.1.34 [16], [27] A topological space (X, τ) is said to be:

(1) $\boldsymbol{T_0}$***-space*** if for any pair of distinct points in X there exists an open set containing one of the points but not the other.

(2) $\boldsymbol{T_{1/2}}$***-space*** if every singleton is either open or closed set.

(3)$\boldsymbol{T_1}$***-space*** if for any pair of distinct points x, y in X there exists O_x such that $y \notin O_x$, and there exists O_y such that $x \notin O_y$.

It is clear that, T_1-space $\Longrightarrow$ $T_{1/2}$-space $\Longrightarrow$ T_0-space.

Definition 1.1.35 [16] Let (X, τ) and (Y, η) be topological spaces. A function f from X into Y is ***continuous*** relative to τ and η if the inverse image $f^{-1}(H)$ of every open subset in η is an open subset in τ.

Definition 1.1.36 [16] Two topological spaces X and Y are called ***homeomorphic*** or topologically equivalent if there exists a bijective function $f: X \longrightarrow Y$ such that f and f^{-1} are continuous.

Definition 1.1.37 [16] The ***Cartesian product*** of an indexed class of sets, $\{A_i \mid i \in I\}$, denoted by $\prod_{i \in I} A_i$, is the set of all functions $p: I \to \bigcup_i A_i$ such that $p(i) = a_i \in A_i$. We denoted such an element of Cartesian product by $p = \langle a_i; i \in I \rangle$. For each $i_0 \in I$ there exists a function π_0, called the i_0-th projection function, from the product set $\prod_{i \in I} A_i$ into the i_0-th coordinate set A_{i_0} defined by $\pi_0(\langle a_i; i \in I \rangle) = a_{i_0}$.

Definition 1.1.38 [16] Let (X_i, τ_i) be a collection of topological spaces and let X be the Cartesian product of the sets X_i ; $X = \prod_i X_i$. The coarsest topology τ on X with respect to which all the projections $\pi_i: X \to X_i$ are continuous is called product topology. The product set X with the product topology τ is called the ***product space***.

Proposition 1.1.6 [16] Let $X_1, \dots, X_m$ be a finite number of topological spaces and let X be the product space, i.e. $X = X_1 \times \dots \times X_m$. Then the following subsets of the product space X, $G_1 \times \dots \times G_m$ where G_i is an open subset of X_i, form a base for the product topology on X.

Corollary 1.1.1 Let β_i be a basis for the topological spaces (X_i, τ_i) ;$i = 1, \dots, m$. Then $\beta = \prod_i \beta_i$ forms a base for the product space $X = \prod_i X_i$.

Proposition 1.1.7 [10] Let X and Y be *Alexandroff* spaces, with smallest basesβ and γ respectively. Then :

(1) If X is a subspace of Y, then $\beta = \{V \cap X \mid V \in \gamma\}$.

(2) $X \times Y$ is an Alexandroff space and has a smallest basis

$$\beta \times \gamma = \{B \times V \mid B \in \beta, V \in \gamma\}.$$

Definition 1.1.39 [16] Two subsets A and B of a topological space (X, τ) are said to be ***separated*** if and only if $A \cap Cl(B) = \phi$ and $Cl(A) \cap B = \phi$.

Definition 1.1.40 [16] A subset A of a topological space (X, τ) is called ***disconnected*** set if there exist open subsets G, H such that $G \cap A$ and $H \cap A$ are disjoint non-empty sets whose union is A. A set is ***connected*** if it is not disconnected.

Theorem 1.1.6 [16] A subset of a topological space is connected if and only if it is not a union of two non-empty separated sets.

Theorem 1.1.7 [16] A topological space (X, τ) is connected iff:

(i) X is not a union of two non-empty disjoint open sets, or equivalently,

(ii) The only subsets of X which are both open and closed sets are X and ϕ.

Theorem 1.1.8 [16] Let A be a subset of a topological space (X, τ). Then A is connected with respect to τ iff A is connected with respect to the relative topology τ_A on A.

Definition 1.1.41 [21] Let X be a connected space and $x \in X$. Then x is called a ***cut point*** if $X \setminus \{x\}$ is disconnected.

Definition 1.1.42 [9] A topological space (X, τ) is called ***extremely disconnected*** if for every open set $G \subseteq X$ the closure $Cl(G)$ is an open set in X.

Definition 1.1.43 [16] A ***connected component*** E of a topological space (X, τ) is a maximal connected subset of X.

Definition 1.1.44 [16] Let $I = [0,1]$, the closed unit interval. A ***path*** from a point a to a point b in a topological space (X, τ) is a continuous function $f: I \to X$ with $f(0) = a$ and $f(1) = b$. Here a is called the initial point and b is called the terminal point of the path.

Definition 1.1.45 [16] A subset E of a topological space (X, τ) is said to be ***arcwise connected*** if for any two points $a, b \in E$ there is a path $f: I \to X$ from a to b which is contained in E, i.e., $f(I) \subseteq E$. The maximal arcwise connected subset of X is called ***arcwise connected component***.

Theorem 1.1.9 [16] Arcwise connected sets are connected.

The converse of the previous theorem is not true (see [16] chapter 13).

Theorem 1.1.10 [9] Let $\{(G_s, \tau_s)\}_{s \in S}$ be a family of connected subspaces of a topological space(X, τ). If there exists an $s_0 \in S$ such that G_{s_0} is not separated from any of the sets G_s, then the union $\bigcup_{s \in S} G_s$ is connected.

Theorem 1.1.11 [21] Let $\{X_i\}_{i \in I}$ be a non-empty family of non-empty spaces. The product space $\prod X_i$ is connected iff each X_i is connected.

1.2. Digital topology.

The word digital comes from the Latin digitus, meaning "finger or toe". The herb purple foxglove, has flowers that look like a bunch of fingers (or gloves rather) as shown in Figure 1.1. In Latin is called

Digitalis purpurea. This herb is the source of medicine that is still today one of the most important drugs for controlling the heart rate [10].

Figure 1.1 shows the herb purple foxglove which is the source of medicine used for controlling the heart rate.

In our context, digital is used to mean that it is possible to count points in the digital space using fingers and toes. Now we state some needed graph-theoretical concepts.

Definition 1.2.1 [32] The ***digital n-space*** $\mathbb{Z}^n$ is the set of n-tuples $x = (x_1, x_2, \dots x_n)$ of the real Euclidean n-space having integer coordinates. The digital (2- or 3-) space is the most commonly used in computer graphics and computer vision. A point with integer coordinates is called a ***digital point*** or grid point. Any subset X of the digital space is called ***digital set***. For example, the digital line is $\mathbb{Z}$, the digital plane is $\mathbb{Z}^2$ and the digital 3-space is $\mathbb{Z}^3$.

Definition 1.2.2 [32] Given a point $x = (x_1, x_2, \dots x_n) \in \mathbb{Z}^n$ the ***$(3^n - 1)$-neighbors*** of x are all points with integer coordinates $y = (y_1, y_2, \dots y_n)$ such that $\max_{i=1,\dots n} |x_i - y_i| = 1$ which is 8-neighbors in the digital plane $\mathbb{Z}^2$ and 26-neighbors in $\mathbb{Z}^3$.