Analysis of the System of Linear Algebraic Equations Arising from the Solution of Some Plane Boundary-Value Problems in a Rectangular Region when using the Boundary Fourier Expansion Method

By

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Arabic Summary I

Abstract.

We have investigated four different systems of linear algebraic equations arising from plane singular boundary-value problems for Laplace's equation in rectangular regions, with the aim of comparing the efficiency of three methods of solution: the QR-Factorization Method contaminated with noise, the Dynamical systems Method and the Variational Regularization Method. In each case, we have studied thoroughly the matrix of coefficients.

We have reached some conclusions concerning the relative efficiency of the three above-mentioned methods of solution. Moreover, it turns out that it is the minimum singular value of the matrix of coefficients, and not the condition number, that intervenes in determining the efficiency of the QR-Factorization Method.

Keywords: Singular Laplace's equation for rectangular regions, Trefftz's method, Dynamical systems method (DSM), Variational regularization method (VRM), QR-factorization technique.

Introduction

This thesis is a contribution to the field of solution methods for linear ill-posed problems. This field has been in the focus of mathematical literature for many years. It occurs in a wide variety of applications such as Geophysics [50], Astrometry [18], Mathematical Biology [19], and Image Restoration ([8], [21]). The notion of a well-posed problem and an ill-posed problem goes back to a famous paper by Jacques Hadamard published in 1902 [30]. In many science and engineering applications it is necessary to compute an approximate solution of the linear system of equations:

$$Ax = b, \qquad A \in R^{m \times n}, \ m \geq n. \tag{*}$$

Throughout this thesis, the right-hand side b is usually corrupted by noise, i.e.:

$$b := b + \delta \ ,$$

where δ accounts for noise and measurement errors. According to Hadamard [30] the problem (*) is said to be well-posed if the following conditions are satisfied: (a) the solution exists, (b) the solution is unique and (c) the solution continuously depends on the data. Otherwise, we say the system (*) is ill-posed system. To determine an approximate solution to the system (*) after adding noise, unfortunately due to the noise on b, the direct solution of (*) is a poor approximation of the true solution x, because the system has a huge condition number and hence A is nearly singular, so the computed solution in this case may be inexact. This is quite unfortunate since b is often affected by noise or measuring errors and computers only have finite floating point precision. When solving the system, the noise and rounding errors can disturb more and more, and the solution can easily end up being completely useless. Many mathematical problems are ill-posed.

The two-dimensional boundary-value problems for Laplace's equation arise in various branches of Science, as useful models for the more realistic, three-dimensional problems. A tremendous number of applications of these problems may be found in the literature, for example in Engineering problems related to nondestructive testing techniques [7,10,11,13,14], in Geophysics and in Biomechanics [17,70], in problems of heat transfer [22, 23]. Such problems may be ill-posed in the sense that a small change in the data may result in a dramatic change in the solution (for example, the famous Hadamard's case for steady heat conduction in a semi-infinite strip). Under an additional a priori boundedness condition, a continuous dependence of the solution on the data can be obtained. This is called conditional stability [53]. Other conditional stability conditions for Laplace's equation may be found in [15, 34]. Due to the severely ill-posedness of such problems, numerical computation becomes quite difficult. In order to obtain a stable numerical solution for such ill-posed problems, many regularization methods have been proposed, among which: The quasi-reversibility method [43,47], the Tikhonov regularization method [9,73], the boundary element method [33], the discretization method [25,65], the moment problem method [16,39], Fourier regularization method [26], the optimal regularization method [70], and others.

The present Thesis is devoted to the study of systems of linear algebraic equations that arise from four plane singular boundary-value problems for Laplace's equation describing steady-state heat conduction in rectangular plates [6, 45]. The use of a modified variant of Trefftz's method for finding approximate solutions ultimately produces rectangular systems of linear algebraic equations. These are characterized by large condition numbers, due to the presence of boundary singularities for the solution functions, in addition to the effect of the corner boundary points. The QR-factorization method was used in [6] to obtain

efficient approximate solutions for the considered boundary-value problems, in the sense that these solutions can be easily computed as finite linear combinations of harmonics, and the resulting error in satisfying the boundary conditions is relatively low as compared to that obtained by the use of the standard Boundary Collocation Method or by the standard Finite Element Method.

It is our purpose here to present and analyze a recent method called dynamical systems method (DSM) for stable solutions of linear ill-posed problems. Also, we present one of the traditional stable methods for solving linear ill-posed problems, called Tikhonov (or variational regularization) method (VRM) . Comparison between the performance of the QR-factorization method contaminated with noise in dealing with such systems of linear algebraic equations, with the two other methods which are known to be stable: The Dynamical Systems Method (DSM) and the Variational Regularization Method (VRM) is one of the main goals of this thesis. The numerical results presented here put in evidence the difficulties that may arise from the use of the QR-factorization method in solving the systems of linear algebraic equations due to instability. We have reached some conclusions concerning the comparison between the above-mentioned methods of solution.

Chapter 1

Some Mathematical Tools

The purpose of this part is to introduce some tools and concepts needed in the thesis. For our purpose, we will give at first some definitions, for instance, Vector Space, Matrix Norm,..., etc. Also we will give some theorems. Almost all results are stated without proofs, but we reference to for more details.

1.1 Basic Notations and Auxiliary Results

Definition 1.1.1 (Vector Space) A vector space over a field K is a nonempty set X of elements x, y, (called vectors) together with two algebraic operations. These operations are called vector addition and multiplication of vectors by scalar. (For more details see [46])

Definition 1.1.2 (Inner Product Space) An inner product space is a vector space X with an inner product defined on X. Also, an inner product on X is a mapping of X×X into the scalar field K of X; that is, with every pair of vectors x and y there is associated a scalar which is written < x, y > and is called the inner product of x and y, and the inner product satisfies:

(i) $< x+y, z > = < x, z > + < y, z >$,

(ii) $< \alpha x, y > = \alpha < x, y >$,

(iii) $< x, y > = \overline{< y, x >}$,

(iv) $< x, x > \geq 0$ and $< x, x > = 0$ iff $x = 0$.

Let X be a vector space over the field $K := C$ of complex numbers or over the filed $K := R$ of real numbers.

Definition 1.1.3 A **norm** is a mapping function, $\|.\| : X \to R$, which for all $x, y \in X$ satisfies the following conditions

(i) $\|x\| \geq 0$ for all $x \in \mathrm{X}$,

(ii) $\|x\| = 0 \leftrightarrow x = 0$,

(iii) $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in \mathrm{K}$,

(iv) $\|x+y\| \leq \|x\| + \|y\|$ (Triangular Inquality).

A useful class of vector norms is the Hölder p-norms, defined by

$$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}, \ p \geq 1.$$

The three most widely used p-norms are the 1, 2 and ∞ norms:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|,$$

$$\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{1/2} = (x^* x)^{1/2},$$

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

The 2-norm, also called the Euclidean norm, has the useful property that it is invariant under unitary transformations. For real matrix Q we have $Q^* Q = I$ and so

$$\|Qx\|_2^2 = x^* Q^* Q \, x = x^* x = \|x\|_2^2.$$

Lemma 1.1.1 (Schwarz Inequality)

An inner product and the corresponding norm satisfies the Schwarz inequality

$$| < x, y > | \leq \|x\| \|y\|,$$

where the equality sign holds if and only if $\{x, y\}$ is a linearly dependent set.

Proof: See [46].

Definition 1.1.4 (Hilbert Space) A Hilbert space is a complete inner product space. An example of Hilbert space is an Euclidean space R^n with inner product defined by

$$< x, y > = \xi_1\eta_1 + \ldots + \xi_n\eta_n ,$$

where $x = (\xi_i) = (\xi_1, \cdots, \xi_n)$ and $y = (\eta_i) = (\eta_1, \cdots, \eta_n)$.

Definition 1.1.5 (Linear Operator) A linear operator T is an operator such that

(i) the domain $D(T)$ of T is a vector space and the range $R(T)$ lies in a vector space over the same field.

(ii) for all $x, y \in D(T)$ and scalar α ,

$$T(x+y) = Tx + Ty,$$
$$T(\alpha x) = \alpha T x.$$

Definition 1.1.6 (Bounded Linear Operator) Let X and Y be normed spaces and $T : D(T) \to Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be bounded if there is a real number c such that

for all $x \in D(T)$,

$$\| Tx \| \leq c \| x \| .$$

Definition 1.1.7 (Hilbert-adjoint Operator T^*) Let $T : H_1 \to H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then the Hilbert-adjoint operator T^* of T is the operator

$$T^* : H_2 \to H_1$$

such that for all $x \in H_1$ and $y \in H_2$,

$$< Tx, y > = < x, T^* y > .$$

Theorem 1.1.1 The Hilbert-adjoint operator T^* of T in definition (1.1.7) exists, is unique and is a bounded linear operator with norm

$$\| T^* \| = \| T \| .$$

Proof. See [46].

Definition 1.1.8 (Self-adjoint Operator) A bounded linear operator $T: H \to H$ on a Hilbert space H is said to be self-adjoint or Hermitian if $T^* = T$.

Note that: The Hilbert-adjoint operator T^* of T is defined by

$$< T x, y > = < x, T^* y > .$$

If T is self-adjoint operator, the formula becomes

$$< T x, y > = < x, T y > .$$

See [46].

Definition 1.1.9 (Dense Set) A subset M of a metric space X is said to be dense in X if

$$\overline{M} = X ,$$

where $\overline{M}$ is the closure of M.

Remark 1.1.1 The operator T is called densely defined in H, when $D(T)$ is dense in H.

1.2 Matrix and Matrix Norms

•An $n \times n$ matrix A is said to be nonsingular if A^{-1} exists, with $AA^{-1} = A^{-1}A = I$. If matrix A^{-1} does not exist then A is called singular (i.e. $\det(A) = 0$).

•A $n \times n$ real matrix A is said to be symmetric if $A = A^*$, where A^* is the transpose of A.

•An $n \times n$ matrix A is said to be positive definite if $x^* A x > 0$ for all nonzero $x \in R^n$.

•An $n \times n$ matrix A is said to be positive semi-definite if $x^* A x \geq 0$ for all nonzero $x \in R^n$ (or equivalently, all eigenvalues are nonnegative).

• A real square matrix Q is orthogonal if $Q^* Q = Q Q^* = I$.

• The rank of a matrix is the dimension of the column space of A. It is denoted by $\text{rank}(A)$.

• An $n \times n$ matrix A is said to have full column rank if its columns are linearly independent. The full row rank is similarly defined.

• A is said to have full rank if it has either full column rank or full row rank.

• If A does not have full rank, it is called rank deficient.

Definition 1.2.1 A **matrix norm** is a function that satisfies the following conditions (see [12] and [46]):

(i) $\|A\| \geq 0$ for all nonzero $A \in R^{m \times n}$,

(ii) $\|\alpha A\| = |\alpha| \|A\|$ for all $\alpha \in \mathrm{R}$, $A \in R^{m \times n}$,

(iii) $\|A+B\| \leq \|A\| + \|B\|$ for all $A, B \in R^{m \times n}$ (Triangular Inquality),

(iv) $\|A B\| \leq \|A\| \ \|B\|$ for all $A \in R^{m \times n}$, $B \in R^{n \times m}$.

The most commonly used matrix norm is the Frobenius norm, defined by

$$\|A\|_F = (\sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|^2)^{1/2} = (trace(A^* A))^{1/2} \ ,$$

where $trace(A)$ is defined as the sum of the diagonal entries of A. Given a vector norm, the corresponding subordinate matrix norm is given by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| \ .$$

Just as the case for vectors, the three most common subordinate matrix norms are the 1, 2 and 1 norms:

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|,$$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|,$$

$$\|A\|_2 = (\rho(A^* A))^{1/2},$$

where the spectral radius:

$$\rho(A) = \max\{|\lambda| : \det(A - \lambda I) = 0\}.$$

1.3 The Least Squares Problem (LSP)

Consider the linear system of equations:

$$Ax = b \quad ,$$

then the least squares problem (LSP) is to find x such that:

$$\min_{x \in R^n} \| Ax - b \|_2 ,$$

where $A \in R^{m \times n}$, $m \geq n$ and $b \in R^m$.

If $m > n$, the problem is called an overdetermined LSP, if $m < n$, it is called an underdetermined problem (see [20], [28] and [48]).

Theorem 1.3.1 (Existence and Uniqueness of LSP)

The least-squares solution to $Ax = b$ always exists. The solution is unique if and only if A has full rank. Otherwise, it has infinitely many solutions. The unique solution x is obtained by solving $A^* Ax = A^* b$.

Proof: See [20].

1.4 The Singular Value Decomposition (SVD)

Definition 1.4.1 Let A be a real $m \times n$ matrix, $m \geq n$. A decomposition of the form

$$A = U \Sigma V^*,$$

where $U = (u_1, \ldots, u_m) \in R^{m \times m}$ and $V = (v_1, \ldots, v_n) \in R^{n \times n}$ are orthogonal matrices and the $m \times n$ matrix $\Sigma = diag(\sigma_1, \ldots, \sigma_n)$ is a diagonal matrix. Such a decomposition is called singular value decomposition (SVD) of A.

Remark 1.4.1 The diagonal entries of Σ are called the singular values of A. The columns of U are called the left singular vectors, and those of V are called the right singular vectors. The singular values are unique, but U and V are not unique. Also