Ain Shams University
Faculty of Science
Department of Mathematics

On Extensions of Filtered and Graded Rings

A Thesis

Submitted in Partial Fulfillment for
the Requirements of the Degree of Master of Science
(Pure Mathematics)

By

Ahmed Gaber Hanafy Mahmoud
Department of Mathematics, Faculty of Science,
Ain Shams University

Supervised by

Prof. Dr.
Abdel-Aziz El-Azab Rdwan
Head of the Department of
Mathematics, Faculty of science,
Ain Shams University

Prof. Dr.
Salah El-Din Sayed Hussein
Department of Mathematics,
Faculty of Science,
Ain Shams University

TO
Department of Mathematics
Faculty of science,
Ain Shams University
Cairo, Egypt
2012

Acknowledgment

ACKNOWLEDGEMENT

I humbly acknowledge the blessings of **Almighty**, **Compeller** and **Subdeur Allah**, without **Whose** mercy and guidance this work never has been started nor completed. May **Allah** prays on **Mohamed** "Peace Be Upon Him" the Prophet and the Messenger of **Allah**.

I would like to express my deep gratitude to **Prof. Dr. Abdel-Aziz El-Azab A. Radwan**, Department of Mathematics, Faculty of Science, Ain Shams University for his patience, criticism, advice and much help. I am grateful for his careful review of the manuscript. I do not forget his encouragement, advice in reading the background material for this work and for giving so generously of his time to complete this thesis in final form.

I wish to acknowledge my thanks to **Prof. Dr. Salah El-Din Sayed Hussien,** Professor of Pure Mathematics, Department of Mathematics, Faculty of Science, Ain Shams University;   for his encouragement, advice and comments.

I wish to record my gratitude to the staff of Department of Mathematics, Faculty of Science, Ain Shams University, for facilities extended to me and giving me opportunity of further my education.

Many thanks also go to all my colleagues, in my department whom wished for me a good luck, wishing for them all the best in the life.

To my family, I offer my sincere gratitude for their love and support and giving me opportunity of further my education.

CONTENTS

Abstract

Abstract

This thesis is devoted to the study of filtered and graded Procesi extensions of filtered and graded rings. After a brief study of general features concerning filtered and graded Procesi extensions at the level of graded Rees rings and their associated graded rings, we turn to the behavior of filtered and graded Procesi extensions towards the micro-affine schemes. We show that these extensions behave well from the geometric point of view.

Summary

Summary

The extensions of rings provide a useful tool for obtaining algebraic theorems and results and additional structures for both rings, prime ideals of both rings, modules over both rings with some functors and algebras over both rings. The second level of prime ideals; in general of ideal theory, occupies an important place in the algebraic geometry.

Ring extensions, like field extensions, can be considered from two points of view. One can look upward from a ring to its extensions or downward to its subrings. The work in this thesis provides an example of the upward point of view.

There are many different ways of constructing the ring extensions and their applications; several of these were thought to be different in some papers, see [5], [9], [11], [14], [17], [18], [25] and [26].

Graded ring extensions permit us to construct structure sheaves extensions, on the micro-structure sheaves, on the graded prime spectrum $Spec^g(G(R))$ of the associated graded ring $G(R)$, when this space is endowed with the Zariski graded topology.

This thesis is devoted to the study of filtered and graded Procesi extensions of filtered and graded rings. After a

brief study of general features concerning filtered and graded Procesi extensions at the level of graded Rees rings and their associated graded rings, we turn to the behavior of filtered and graded Procesi extensions towards the micro-affine schemes. We show that these extensions behave well from the geometric point of view.

The thesis consists of three chapters :

Chapter 1 : Preliminaries

This chapter provides the preliminaries and the back ground material to be used in subsequent chapters. We provide a brief survey of the basic definitions and elementary results concerning extensions of a ring, prime spectrum of a ring, graded rings and filtered rings as well as sheaves and schemes.

Chapter 2 : Filtered and Graded Procesi Extensions of Rings

In this chapter we continue the study of filtered and graded Procesi extensions of filtered and graded rings introduced in [17].

In the first section of this chapter, we define a new filtration $F''S$ of S: $F''S = \{F''_nS\}_{n\in Z}$; with $F''_nS = \varphi(F_nR).S^R$. As in [17], we study, over the filtered level, the passage of various ring theoretic properties between R, S and concern with the relationship between the filtration $F''S$ on S and those studied in [17].

In the second section, with respect to $F''S$, we prove that $G(S)$ is graded Procesi extension of $G(R)$ and, for any $n \in Z$, $\tilde{S}/X^n\tilde{S} = \bar{\tilde{S}}(n)$ is a graded Procesi extension of $\bar{\tilde{R}}$ as in Proposition 2.2.5.

Chapter 3 : Procesi Extensions of Filtered and Graded Rings Applied to the Micro-Affine Schemes

In this chapter we turn to the behavior and graded Procesi extensions towards to the micro-affine schemes . A number of results concerning these concepts are given.

In the first two sections we introduce a survey, sometimes with proofs, on the graded spectrum of the associated graded ring $G(R)$ of R. Some results about the micro-affine schemes are concerned, see [6], [10], [15], [21] and [26]. This survey represents a solid foundation for our results in this chapter.

In the third section we prove that the filtered Procesi extension $\varphi : R \to S$, for every $n \in Z$, induces a graded Procesi extension

$$(Y = Spec^g(G(S)), \underline{\tilde{O}}_Y^{(n)}) \longrightarrow (X = Spec^g(G(R)), \underline{\tilde{O}}_X^{(n)})$$

of affine schemes.

By considering the inverse limit in the graded sense and the idea of micro-affine structure sheaves, we pay our attention to deduce that

$$(Y, \underline{O}_Y^{\mu}) \to (X, \underline{O}_X^{\mu})$$

, of filtered micro-affine schemes, is a filtered Procesi extension.

The main results of Chapters 2 and 3 seem to be original and have been published in the International Mathematical Forum (Bulgaria), Vol.7, 2012, no. 26, 1279 -1288.

Chapter 1

Chapter 1

PRELIMINARIES

In this chapter we introduce the background material which we need in this thesis. However, we just provide a brief survey of the basic definitions and elementary results concerning extension of a ring, prime spectrum of a ring , graded rings, filtered rings as well as sheaves and schemes.

1.1 Extensions of a Ring

Definition 1.1.1. [14]
Let R, S be rings and $\varphi : R \to S$ a ring homomorphism. Then, following **C. Procesi** [see [14]], we say that φ is an **extension** of rings if $S = \varphi(R).S^R$, where

$$S^R = \{s \in S; s\varphi(r) = \varphi(r)s \forall r \in R\}.$$

Examples 1.1.2.
1.If $\varphi : R \to S$ is an epimorphism and S is commutative then φ is an extension (in the sense of Procesi) of rings.

2.Assume that φ is an involution of a field K and put $S = K[x, \varphi]$, the skew polynomial ring over K (we write the coefficients in the right). Then S is an extension of K called strongly normalizing extension.

1.2 The Prime Spectrum of a Ring

This section is mainly concerned with the most fundamental properties of the prime spectrum of a commutative ring with unity. Questions posed by M. Atiyah and I. Macdonald in their book (Introduction to Commutative Algebra), serve as a guideline through most of this section.

Let R be a commutative ring with unity and let X denote the set of all prime ideals of R. Our goal is to endow X with a topology. To this end, for each subset $E \subset R$, let $V(E) = \{P \in X; E \subset P\}$. Suppose I is the ideal generated by $E \subset R$. Since $E \subset I \subset r(I)$, where $r(I)$ denotes the radical of I, we clearly have $V(\langle I \rangle) \subset V(I) \subset V(E)$. However, I is the smallest ideal containing E so that:
$P \in V(E) \Rightarrow P \in V(I)$.

Hence, $V(E) = V(I)$. Also, $r(I)$ equals the intersection of all prime ideals containing I so that $V(r(I)) = V(I)$. Therefore, for any subset $E \subset R$ we have:
$V(E) = V(I) = V(r\langle I \rangle)$.

Consider the cases when $E = \{0\}$ or $E = \{1\}$. Since