Ain Shams University
Faculty of Education
Mathematics Department

ON INTERVAL VALUED FUZZY TOPOLOGICAL SPACES

A Thesis Presented By

Salem Ali Mohammed Saleh

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SUPERVISORS

Prof. Dr. Ali Kandil Saad Ibrahim
Prof. of Pure Mathematics, Faculty of Science, Helwan University
Prof. Dr. Osama A. El-Tantawy
Prof. of Pure Mathematics, Faculty of Science, Zagazig University
Dr. Mohammed Mostafa Yakout
Lecturer of Pure Math, Faculty of Education, Ain Shams University.

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Ain Shams University
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Approval Sheet

Candidate: **Salem Ali Mohammed Saleh**

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Supervisors:

Name	Signature
Prof. Ali .Kandil Saad Ibrahim Prof. of Pure Mathematics, Faculty of Science, Helwan Univ.	
Prof. Osama A. El-Tantawy Prof. of Pure Mathematics Faculty of Science, Zagazig Univ.	
Dr. M. M. Yakout Mousa Lecturer of Pure Mathematics, Faculty of Education, Ain Shams Univ.	

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CONTENTS

Chapter IV: Separation Axioms in Interval Valued Fuzzy Topological Spaces

Chapter V: C*-Compactness in L-fuzzy Topological Spaces

Chapter VI: On Fuzzy Topological Subspaces

List of Notations

Fuzzy Sets……………………………………………............FSs

L-Fuzzy sets ………………………………………...............L-FSs

L-Fuzzy Topological Spaces.......................................L-FTSs

Interval Valued Fuzzy Sets …………………..............IVFSs

Interval Valued Fuzzy Topological Spaces…..............IVFTSs

Interval Valued Fuzzy Open Sets…… ....IVFOSs

Interval Valued Fuzzy Closed Sets…………….............IVFCSs

Interval Valued Fuzzy product……………………. $\hat{\times}$-Product

Interval Valued Fuzzy closure operator......IVF-closure operator

Interval Valued Fuzzy interior operator....IVF-interior operator

Interval Valued Fuzzy Continuous … ....IVF-continuous

Interval Valued Fuzzy Separation……............. IVF-separation.

Interval Valued Fuzzy Regularity…….................IVF-Regularity

Note: In general the notation "IVF" denotes to the concept of " Interval Valued Fuzzy "

Abstract

The objective of this thesis is to introduce and study the regularity and separation axioms in some topological categories such as L-fuzzy topological spaces, interval valued fuzzy topological spaces and fuzzy topological subspaces (fuzzy topology on a fuzzy set). It contains the theoretical results on the development of theory of lattice and interval valued fuzzy topology. We begin by a survey of some natural extensions of fuzzy set theory such as L-fuzzy sets, interval valued fuzzy sets. We reform the concept of interval valued fuzzy sets in more convenient form, examining many of its fundamental properties. We recalled and introduced some important concepts concerning theory of lattice and interval valued fuzzy topological spaces. After that we introduced a new set of regularity and separation axioms in L-fuzzy topological spaces, then extend this to the interval valued fuzzy case, fuzzy topology on fuzzy set case and obtain analogous results and characterizations to those of the fuzzy case, investigating some topological concepts in these approaches. We then briefly examine the compactness in L-fuzzy topological spaces, we gave a new notion of the cover called P-cover, based on the fuzzy points of the fuzzy set, consequently, we introduce another type of compactness in L-fuzzy topological space called C*-compactness which is avoid many deviations in other sequels, investigating many interesting properties in this sequel.

Introduction

L. Zadeh[113] introduced the theory of fuzzy sets in (1965). From the beginning it was clear that this theory was an extraordinary tool for representing human knowledge. Nevertheless, L. Zadeh [116] himself established in (1973) that sometimes, in decision-making processes, knowledge is better represented by means of some generalizations of fuzzy sets. The so-called extensions of fuzzy set theory arise in this way such as, L-fuzzy sets [44], Interval valued fuzzy sets [94], Vague set [108], Fuzzy rough sets [26], Grey sets [20], rough sets[91], Intuitionistic fuzzy sets [4,5] and flou sets [67]. For these extensions, many mathematicians studied the relations between them see [11,18,19,23,59]. Kerre etal. [23] showed the links between interval valued fuzzy sets and other theories modeling imprecision as: Fuzzy rough sets $\rightarrow$ Interval valued fuzzy sets $\Leftrightarrow$ Intuitionistic fuzzy sets $\Leftrightarrow$ L*-fuzzy sets $\Leftrightarrow$ Vague sets $\Leftrightarrow$ Grey sets. In the applied field, the success of the use of fuzzy set theory depends on the choice of the membership function that we make. However, there are applications in which experts do not have precise knowledge of the function that should be taken. In these cases, it is appropriate to represent the membership degree of each element to the fuzzy set by means of an interval. From these considerations the extension of fuzzy sets called theory of Interval-Valued Fuzzy Sets arises that is, fuzzy sets such that the membership degree of each element of the fuzzy set is given by a closed subinterval of the interval [0,1]. Hence, not only vagueness (lack of sharp class boundaries), but also a feature of uncertainty (lack of information) can be addressed intuitively. The theory of interval valued fuzzy set was originally proposed by [94,116] as a generalization of the fuzzy set theory but it is generally attributed to Gorzalczany [47] and Turksen [97] and has been regarded as an important mathematical tool to deal with vagueness. The theory of interval valued fuzzy set or Type-two fuzzy set [58] also has been developed in different aspects which are covers the following topics such as:

- Interval-valued Fuzzy Theory,
- Interval-valued Fuzzy Measures,
- Interval-valued Fuzzy Correlation Measures,
- Interval-valued Fuzzy Logic Systems,
- Operators on Interval-valued Fuzzy Sets.

and their applications to:

- Image Processing,
- Approximate Reasoning,
- Control Systems,
- Medical Diagnosis,
- Classification Models.

For more details see [9-10,21-22,24-26,39,46-51,66,81,84,86,88,90, 97-98,100,104-105,109,115-116].

The topological structures on lattices of fuzzy sets were first introduced in (1968) by Chang, later Chang's ideas were developed in essentially different directions such as [45,55,77,83].

The notion of interval valued fuzzy topology, was first introduced in (1999) by Nanda-Semanta [88]. Nevertheless, separation axioms in interval valued fuzzy topological spaces are not yet studied.

For all the above considerations, we choose studying the topology of interval valued fuzzy sets and the L-fuzzy topological spaces.

The main aims and results of this Thesis can be summarized as:

- Developing and studying the interval valued fuzzy set theory and the theory of interval valued fuzzy topology.
- Establishing and studying a set of separation, regularity axioms in the theory of interval valued fuzzy topology.
- Introducing and investigating a new set of separation axioms in the theory of lattice valued fuzzy topology, by using the concept of quasi-coincident and neighborhood systems.
- Introducing another type of compactness in L-fuzzy topological spaces called C*-compactness which avoids many deviations and possesses many interesting properties.
- Spotting light the notion of fuzzy topology on fuzzy set as one of treatments of the subspace problem in fuzzy topological spaces, and then studying some topological concepts in this approach.

This Thesis includes six chapters as follows:

Chapter I: is a natural introduction, providing the reader with results concerning, lattice theory, fuzzy sets, L-fuzzy sets and interval valued fuzzy sets. It also contains the interrelations between these extensions of the fuzzy set theory.

In Chapter II, first we recalled some concepts concerning lattice valued fuzzy topology which are used in this thesis, we developed the theory of interval valued fuzzy topology, investigating many properties. Based on the concept of IVF-product, we defined some operators such as IVF-closure and IVF-interior, we induce many interval valued fuzzy topologies from a fuzzy topology and vice versa, and we investigated some concepts in IVFTSs such as Cartesian product, initial-IVFT and closure (interior) of Cartesian product of IVFTSs.

Some results of this Chapter are accepted in " An International Journal: Applied Math. and Info. Sci." [72].

In Chapter III, we introduce a set of new separation and regularity axioms so-called $L\text{-}\boldsymbol{FT}_i$ $(L\text{-}\boldsymbol{FR}_i)$, respectively in L-fuzzy topological spaces, investigating many properties and characteristic theorems of them. Also, good extension and hereditary properties of them are examined. The relation between our separation axioms and those different separation axioms presented by other authors are investigated with some necessary counterexamples.

The results of this Chapter are:

- Presented in International Conference of Mathematics hold in Ain Shams Univ. in 27-30 Dec. (2007).
- Accepted in the J. of Egyptian Math. Society [70].

In Chapter IV, we established and studied the theory of separation axioms and regularity axioms in theory of interval valued fuzzy topology, investigating certain results and giving some implications on them. We introduce the concept of good extension property in this setting. Some interrelations between separation axioms of IVFTS and that of induced FTSs and vice versa are introduced with some necessary counterexamples.

The results of this Chapter are accepted in "An International Journal: Applied Math. and Inf. Sciences" [72].

In Chapter V, first we recall some definitions as survey of compactness in L- fuzzy topological spaces, after that, we give a new notion of the cover called P-cover, based on the fuzzy points of the fuzzy set, and consequently, we introduce another type of compactness in L-fuzzy topological space so-called C*-compactness which is avoid many deviations, investigating many interesting properties in this sequel. Some interrelations between the compactness in L-FTSs and that of its induced topologies are studied. The concepts of C*-Lindelof and counatbly C*-compact are intrduced and studied with some properties of them in L-FTSs.

The results in this Chapter are:

- Presented in the 23th Conference of topology and its applications hold in Asuit Univ. in 22-23 Apr. (2009).
- Also, submitted to: "An int. J. of fuzzy logic and intelligent systems" [74].

In Chapter VI, we spotted light on the notion of fuzzy topology on fuzzy sets which is considered as one of treatments of the subspace problem in fuzzy topological spaces, investigating some topological concepts in this approach such as separation, regularity axioms, also we generate some ordinary topologies from a given fuzzy topology on a fuzzy sets A and vice versa. We introduced the definition of lower semi-continuous function in this new setting and then we studied the good extension and hereditary properties.

The results in this Chapter are:

- Presented in the 22th Conference of topology and its applications hold in Helwan Univ. in 7-8 July (2008).
- Also, submitted to" An international Journal: Knowledge-Based Systems "[73].

Chapter I

Preliminaries

Lattice theory plays an important role in many different aspects of mathematics. The concept of a fuzzy set was introduced by Zadeh in [113]. Subsequently, Goguen in [44] suggested the more general theory of L-fuzzy sets. The concept of interval valued fuzzy sets was introduced by Zadeh [116] and Sambuc in [94].

The purpose of this chapter is to give a short survey of some needed definitions and theories of the material used in this thesis. It contains basic concepts of lattice theory, fuzzy sets, L-fuzzy sets and interval valued fuzzy sets theory. It also contains the relation between these extensions of fuzzy set theory. Note that for concepts and results that are used but not stated here we refer to [3-5, 8-11, 14, 16, 18-22, 24-28, 39-59, 66-70, 93, 97-117].

1.1 Lattice Theory

1.1.1 Definition [8]

Let L be a set. A partial order on L is a binary relation $\leq$ which is

1. reflexive: $a \leq a \quad \forall\, a \in L$.

2. transitive: if $a \leq b$ and $b \leq c$ then $a \leq c$, $\forall\, a, b, c \in L$.

3. antisymmetric: if $a \leq b$ and $b \leq a$ then $a = b$, $\forall\, a, b \in L$

A poset$(L, \leq)$ is a set equipped with a partial order relation $\leq$.

1.1.2 Definition [8]

Let L be a poset, S a subset of L. We say that an element $a \in L$ is a *join (supermum or least-upper bound) for S and write* $a = \bigvee S$ *or* $a = supS$ *if,*

1. a is an upper bound for S i.e. $s \leq a \quad \forall s \in S$

2. If b satisfies $s \leq b$ for all $s \in S$, then $a \leq b$.

The antisymmetry axiom ensures that the join of S, if it exists, is unique. If $S = \{s, t\}$ we write $s \vee t$ for $\bigvee\{s, t\}$.

1.1.3 Definition [8]

Let L be a poset, S a subset of L. We say that an element $a \in L$ is a *meet (infimum or greatest-lower bound) for S and write* $a = \bigwedge S$ *or* $a = inf\, S$ *if,*

1. a is a lower bound for S i.e. $a \leq s \ \ \forall s \in S$
2. If b satisfies $b \leq s$ for all $s \in S$, then $b \leq a$.

If $S = \{s, t\}$ we write $s \wedge t$ for $\bigwedge\{s, t\}$.

1.1.4 Definition [8]

A lattice is a poset $(L, \leq)$ such that for all $a, b \in L$, $a \vee b \in L$ and $a \wedge b \in L$.

1.1.5 Proposition [8]

Let $(L, \leq)$ be a lattice. Then the following properties are satisfied:

L_1) $a \vee a = a$, $a \wedge a = a \ \forall a \in L$ (idempotent law)

L_2) $a \vee b = b \vee a$, $a \wedge b = b \wedge a \ \forall a, b \in L$ (commutative law)

L_3) $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c \ \forall a, b \in L$ (associative law)

L_4) $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a \ \forall a, b \in L$ (absorption law).

1.1.6 Definition [75]

An algebraic structure $(L, \vee, \wedge)$ consisting of nonvoid set and two binary operations $\vee$ and $\wedge$ on L is called a lattice if $\vee$ and $\wedge$ satisfy the commutative, associative, idempotent and the absorption laws.

The order-dependent definition $(L, \leq)$ and the algebraic structure of lattice$(L, \vee, \wedge)$ are linked together by the relation:

$a \leq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a \quad \forall\, a, b \in L$.

1.1.7 Definition [75]

i) A lattice $(L, \leq)$ is called bounded if there exist a greatest and smallest element of L denoted by 1,0, respctively.

ii) A bounded lattice $(L, \leq)$ is called complemented if, $\forall\, a \in L \;\exists\, x \in L$ such that $a \wedge x = 0$ *and* $a \vee x = 1$. In this case we say that x is a complement of a.

iii) A lattice $(L, \leq)$ is called chain, if for all $a, b \in L$, either $a \leq b$ or $a \geq b$.

1.1.8 Definition [8]

A lattice $(L, \leq)$ is called a complete lattice if every non-empty subset of L has the supermum and the infimum.

1.1.9 Definition [75]

A lattice $(L, \leq)$ is distributive if the following two distributive laws hold within it:

$i)\; a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$ii)\; a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \;\; \forall\, a, b, c \in L$

1.1.10 Definition [82]

A mapping $' : L \longrightarrow L$ is called an order-reversing involution operator on L if it satisfies the following conditions:

i) $\forall\, a, b \in L$, $a \leq b \Longrightarrow b' \leq a'$

ii) $(a')' = a \;\; \forall \;\; a \in L$.

1.1.11 Lemma [75]

A complemented distributive lattice satisfies De Mogan's laws:

$(a \wedge b)' = a' \vee b'$ and $(a \vee b)' = a' \wedge b' \quad \forall\, a, b \in L$.