Ain Shams University

Faculty of Science

On Extensions of Injectivity, Pseudo-Frobenius and Quasi-Frobenius Rings

Thesis submitted by

Amr Kamel Amin Gad Elrab

M. Sc. Degree in Pure Mathematics (Algebra)

For the award of the
Ph.D.Sc. Degree in Pure Mathematics (Algebra)

Supervised by

Prof. Dr. Robert Wisbauer

Mathematisches Institut
Der Heinrich-Heine-Universität
Düsseldorf,
Germany

Prof. Dr. Abd El-Aziz El-Azab

Head Department of Mathematics
Faculty of Science
Ain Shams University
Egypt

Prof. Dr. Salah El Din Sayed Hussein

Department of Mathematics
Faculty of Science
Ain Shams University
Egypt

Dr. Nasr Anwer S. M. Zeyada

Department of Mathematics
Faculty of Science
Cairo University
Egypt

Submitted to

Department of Mathematics

Faculty of Science

Ain Shams University

2012

Summary

Our two-fold main objective of this work is to introduce and investigate thoroughly two new notions; namely those of *rad*-injectivity and almost injectivity. Let M, and N be right R-modules, where R is an associative ring with unity and all R-modules are unitary. M is called *rad*-N-injective if any R-homomorphism $f : K \longrightarrow M$ where K is a submodule of the Jacobson radical $J(N)$ of the module N, extends to N. A right R-module M is called almost injective, if $M = E \oplus K$ where E is injective and K has zero radical. A ring R is called right almost injective, if R_R is almost injective. Examples are given to show that these notions are distinct from those of mininjectivity, simple-injectivity and injectivity. Most of the basic results on injective modules are shown to hold for *rad*-injective modules. For example, the class of *rad*-injective right R-modules is closed under isomorphisms, direct products, finite direct sums, and summands. The classes of V-rings, semilocal rings, pseudo-Frobenius and quasi-Frobenius rings are characterized in terms of *rad*-injective rings and almost injective rings. It is not known whether ring R for which every finitely generated right R-module (cyclic R- module) can be embedded in a free right R-module, is quasi-Frobenius (artinian) or not. In this work we provide a positive answer if we assume in addition that the ring R is *rad*-injective or almost injective or right weak CS.

The new result we have obtained so far are displayed in chapters 3 and 4 of this dissertation.

In what follows, we give a brief coverage of the contents of this work. The thesis consists of four chapters.

In the first chapter, we present the definitions and the basic properties of injective and continuous modules. Hereditary, Noetherian, and V-rings are characterized in terms of injective modules. We survey several characterizations of quasi-Frobenius rings in particular the result of Faith and Walker on quasi-Frobenius rings. The rest of the chapter is concerned with the structure theory of pseudo-Frobenius rings. The material in this chapter is standard and vitally important for our study and have been included to make the presentation self-contained as far as possible.

The second chapter consists of two main sections. In the first section, we display mininjective rings studied in (cf, [45]). A ring R is called right mininjective if every isomorphism

between two simple right ideals is given by left multiplication. It is shown that R is quasi-Frobenius if and only if R is two sided-artinian and two-sided mininjective. In the second, section we draw our attention to simple-injective rings, where according to Harada, (cf, [26]), a module K is called simple-N-injective if, for every submodule L of N, every R-homomorphism $\gamma : L \longrightarrow N$ with $\gamma(L)$ is simple extends to N. Furthermore, it is shown that a ring R is quasi-Frobenius if and only if R is right noetherian and right simple-injective ring such that socle of the module R_R is essential submodule of R_R i.e. $S_r \subseteq^{ess} R_R$. Finally, it is shown that a semiprimary ring is right self-injective if and only if it is simple-injective. We tried to summarize these known results to suit specialists interested in other lines of algebra and also to make the forthcoming presentation of the remaining chapters clearer and adequate.

In the third chapter, we introduce and investigate the notions of rad-injective and almost injective modules. This chapter is divided into three sections. After introducing the notion of rad-injective modules in the first section we proved that the class of rad-injective right R-modules is closed under isomorphisms, direct products, finite direct sums, and summands. We proved that if R is semilocal ring, then every rad-injective right R-module is injective. We show that if R is a semiprimary, right rad-injective ring with $J^2 = 0$, then R is quasi-Frobenius, if R is semilocal and right rad-injective with ACC on right annihilators, then R is quasi-Frobenius. We round off this section by offering an important characterization of QF -rings. Indeed, we prove that R is right artinian, and right rad-injective ring if and only if R is quasi-Frobenius. In the second section of this chapter, we investigate the relation between rad-injectivity and other injectivity properties. We show that if M is right rad-injective, then M is right mininjective and examples are given to show that the converse is not true. Example 3.2.27 shows that the two classes of simple-injective and rad-injective modules are different. The following conditions are equivalent for a ring R: 1. R is quasi-Frobenius. 2. R is a right rad-injective, right noetherian with $S_r \subseteq^{ess} R_R$. 3. R is a right rad-injective, right Goldie ring with $S_r \subseteq^{ess} R_R$. This extends a well known result in case R is simple injective, (cf, [47], Theorem 6.44). Another convenient characterization of QF-rings is provided in Proposition 3.1.22, which states that: R is quasi-Frobenius if and only if R is left perfect, left and right rad-injective. We show that R is a right rad-injective, and right finitely cogenerated ring if and only if R is a right pseudo-Frobenius. We also show that A ring R is a semilocal, right rad-injective with right essential socle if and only if R is pseudo-Frobenius. Recall that a ring R is called a right V-ring if every simple right R-module is injective (cf, [39]). The class of V-rings is properly contained in the class of right GV-rings (rings for which simple singular right R-modules are injective), (cf, [27]). We show that R is a right V-ring if and only if every right R-module is rad-injective. In third section of this chapter, we investigate almost injective modules and rings. In Proposition 3.3.35 the relation between rad-injectivity and almost injectivity has been established. Proposition 3.3.35 is , in fact, a key proposition which enables us to go further and deeper in our study. Actually, We show that if M is a rad-injective module, then M is almost injective, and an example is given to show that the converse is not true. We show that the following statements are equivalent: 1. Every almost injective right R-module is injective. 2. Every almost injective right R-module is quasi-continuous. 3. R is a semilocal ring. Faith and E. A. Walker have shown in (cf, [33], Theorem 13.6.1) that R is quasi-Frobenius if and only if every right projective R-module is injective if and only if every right injective R-module is projective. We extend this result to rad-injective modules as follows: R is quasi-Frobenius if and only if every rad-injective right R-module is projective. We also prove that every projective right R-module is almost injective if and only if $R = E \oplus T$, where E and T are right ideals of R, E_R is Σ-injective (arbitrary direct sums of copies of E_R are injective) and T_R has zero radical. A ring R which is both right Kasch and right mininjective is not necessarily pseudo-Frobenius. Also if R is right Kasch and right simple-injective, we do not know whether R is right pseudo-Frobenius or not. We give a partial answer for this question in (Theorem 3.3.48)which states the following statements are equivalent for a ring R: 1. R is a right pseudo-Frobenius ring. 2. R is a semiperfect, right almost injective ring with $soc(eR) \neq 0$ for each local idempotent e of R. 3. R is a right Kasch, right almost injective ring. 4. R is a right almost injective ring and the dual of every simple left R-module is simple. One of our main results is Proposition 3.3.50 in which we proved that the following statements are equivalent for a ring R: 1. R is quasi-Frobenius. 2. R is left perfect, left and right rad-injective ring. 3. R is left perfect, left and right almost injective ring. This Theorem extends a well-known result of B. Osofsky (cf, [48]) on self-injective rings. It is not known whether right CF-rings (FGF-rings) are right artinian (quasi-Frobenius). In this section we provide a positive answer as follows: the following statements are equivalent for a ring R: 1. R is quasi-Frobenius. 2. R is right CF and right rad-injective. 3. R is right CF and right almost injective.

In chapter four, we study *soc*-injectivity and (strongly) *soc*-injectivity of modules and rings. This chapter is divided into two sections. In the first section of this chapter we investigate *soc*-injective , *soc*-$C1$ *soc*-$C2$, and *soc*-$C3$ modules. We introduce new properties of *soc*-injective, *soc*-$C2$, *soc*-$C3$ and $CESS$-module. We show that if M_1 and M_2 are right R-modules and $M = M_1 \oplus M_2$. Then M_1 is soc-M_2 injective if and only if for every semisimple submodule S of M such that $S \cap M_1 = 0$, there exists a submodule A of M such that $M = M_1 \oplus A$ and $S \subseteq A$. We show that if M is a $CESS$-module. Then 1. If $M/soc\,(M)$ is finite dimensional, then $M = K \oplus S$ where K is finite dimensional and S is semisimple. 2. If $M/soc\,(M)$ is noetherian, then $M = K \oplus S$ where K is noetherian and S is semisimple. 3. If $M/soc\,(M)$ is artinian, then $M = K \oplus S$ where K is artinian and S is semisimple. In the second section of the chapter we show that the following statements are equivalent for a ring R: 1. R is a right pseudo-Frobenius-ring. 2. R is a semiperfect, right rad-injective ring with $soc(eR) \neq 0$ for each local idempotent e of R. 3. R is a right finitely cogenerated, right rad-injective ring. 4. R is a right Kasch, right rad-injective ring. 5. R is a right rad-injective ring and the dual of every simple left R-module is simple. Gómez Pardo and Guil Asensio (cf, [21]) proved that CF rings are artinian in case R is a CS. We extend this results in Theorem 4.2.52 as follows: if R is a right weak CS and right CF ring then R is right artinian. Gómez Pardo and Guil Asensio (cf, [20]) proved that FGF rings are quasi-Frobenius in case R is a CS. We conclude by strengthening this result in Theorem 4.2.54 which states that if R is a right FGF a right weak CS ring, then R is quasi-Frobenius.

Throughout this thesis all rings considered are associative with unity and all R-modules are unitary.

Two papers have been extracted from this dissertation, namely [61] and [57].

List of Symbols

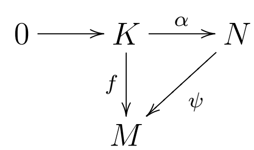
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Rings of integers
$\mathbb{Q}$	Field of rational numbers
$\mathbb{Z}_n$	Ring of integers modulo n
$\lvert X \rvert$	Cardinality of a set X
$E\,(M)$	Injective hull of the module M
$soc\,(M)$	Socle of the module M
$J\,(M)$	Jacabson radical of the module M
$dim\,(M)$	Uniform (Goldie) dimension of the module M
$Z\,(M)$	Singular submodule of the module M
$S_r,\, S_l$	$soc\,(R_R),\, soc\,({}_RR)$
$Z_r,\, Z_l$	$Z\,(R_R),\, Z\,({}_RR)$
$Z_2^r,\, Z_2^l$	$Z\,(R/Z_r) = Z_2^r/Z_r,\, Z\,(R/Z_l) = Z_2^l/Z_l$
$J,\, J\,(R)$	Jacobson radical of the ring R
$r\,(X),\, l\,(X)$	Left and right annihilators of the set X
$M_n\,(R)$	Ring of $n \times n$ matrices over the ring R
$end\,(M)$	Endomorphism ring of the module M
$K \subseteq^{ess} M$	K is essential submodule of the module M
$K \subseteq^{max} M$	K is maximal submodule of the module M
$K \subseteq^{\oplus} M$	K is a direct summand of the module M
$M^{(I)}$	The direct sum of $\lvert I \rvert$ copies of the module M
M^I	The direct product of $\lvert I \rvert$ copies of the module M
M^*	Dual of the module M i.e. $M^* = Hom(M, R)$
$modR$,	Category of right modules over the ring R

Chapter 1. Preliminaries

In this chapter, we assemble basic concepts and associated necessary results for our study in this thesis. We are going to present these concepts in a more detailed fashion, so that the presentation of the thesis becomes more or less self-contained. Throughout this chapter R is an associative with unity and all modules are right R-modules. For further related results, the reader is kindly referred to [2], [12], [15], [35], and [40].

1.1. Injective and Continuous Modules (cf, [5], [6], [7], [11] and [47])

Let M_R, L_R and N_R be right R-modules. M is called *N-injective* if, for any submodule K of N, any R-homomorphism $f : K \longrightarrow M$ can be extended to N. Equivalently, for each R-monomorphism $\alpha : K \to N$ and each R-homomorphism $f : K \to M$ there is an R-homomorphism $\psi : N \to M$ such that the diagram



commutes i.e., $f = \psi\alpha$.

A module M is called *quasi-injective* if M is M-injective. M is called *injective*, if M is *N-injective* for all right R-modules N. The ring R is called *right (self-)injective*, if the right R-module R_R is injective.

Let A be submodules of a module M. Then a submodule C of M is called *a complement* of A in M if it is maximal with respect to $A \cap C = 0$. Such submodule C always exist, by virtue of Zorns Lemma; in fact, any submodule C_1 of M satisfying $A \cap C_1 = 0$ can be enlarged to a complement of A.

A submodule A of a module M is called *essential* in M or M is essential extension of A (denoted by $A \subseteq^{ess} M$ or $A \trianglelefteq M$) if for every submodule $B \subseteq M, A \cap B = 0$ implies $B = 0$.

A submodule C of a module M is said to be *a closed submodule* of M if C has no proper essential extensions inside M.

If M is a *quasi-injective module* then M satisfies the following conditions (cf, [30] and [32]):

1. $C1$ condition: every submodule of M is essential in a summand

(a module satisfying this condition is called a CS-module or extending module).

2. $C2$ condition: if K and L are submodules of M, $K \cong L$, and

$K \subseteq^{\oplus} M$, then $L \subseteq^{\oplus} M$.

3. $C3$ condition: if K and L are submodules of M with $K \cap L = 0$, $K \subseteq^{\oplus} M$ and $L \subseteq^{\oplus} M$, then $K \oplus L$ is a summand of M.

A right R module M is called continuous if M satisfies both $C1$ and $C2$; whereas M is called quasi-continuous or *(π-injective module)* if M satisfies both $C1$ and $C3$. Obviously, every injective or quasi-injective or semisimple module is both continuous and quasi-continuous.

We have just seen that the following implications hold:

Injective $\Rightarrow$ quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ $C1$.

In the sequel of this section, we recall the basic properties of injective, continuous and quasi-continuous modules. Hereditary, Noetherian, and V-rings are characterized in terms of injective modules.

Proposition 1.1.1. The following statements are true:

1. Let N be a right R-module and $\{M_i : i \in I\}$ a family of right R-modules. Then the direct product $M = \prod_{i \in I} M_i$ is N-injective if and only if each M_i is N-injective,$i \in I$.

2. Let M, N, and K be right R-modules with $K \subseteq N$. If M is N-injective, then M is both K-injective and N/K-injective.

3. Let N be a right R-module and $\{A_i : i \in I\}$ a family of right R-modules. Then N is $\bigoplus_{i \in I} A_i$-injective if and only if N is A_i-injective, $\forall i \in I$.

4. If A, B, and M are right R-modules, $A_R \cong B_R$, and M is A-injective, then M is B-injective.

An additive abelian group G is called divisible if $nG = G$ for any $0 \neq n \in \mathbb{Z}$. For example $\mathbb{Q}$ and the Prüfer group $\mathbb{Z}_{p^\infty}$ for any prime p are both divisible. It is known that a right $\mathbb{Z}$-module M is injective if and only if M is divisible.

Definition 1.1.2. A module E is called the injective hull of a module M, denoted by $E(M)$, if E is an essential extension of M and E is injective.

Lemma 1.1.3. If $M = \oplus_{i=1}^{n} M_i$ is a finite direct sum of modules, then $E\left(\oplus_{i=1}^{n} M_i\right) = \oplus_{i=1}^{n} E\left(M_i\right)$.

The following Lemma gives a characterization of relative injectivity in terms of injective hull.

Lemma 1.1.4. A module G is M-injective if and only if $\lambda(M) \subseteq G$ for all R-linear maps $\lambda : E(M) \longrightarrow E(G)$.

Lemma 1.1.5. A module M is quasi-injective if and only if M is fully invariant in its injective hull $E(M)$.

Corollary 1.1.6. Let M be a quasi-injective module. If $E(M) = \oplus_{i \in I} K_i$. Then $M = \oplus_{i \in I}(M \cap K_i)$.

Lemma 1.1.7. **(Baer Criterion)**. A right R-module E is injective if and only if, whenever $T \subseteq R$ is a right ideal, every map $\gamma : T \to E$ extends to $R \to E$, that is, $\gamma = c\cdot$ is multiplication by an element $c \in E$.

Remark 1.1.8. A module P is projective if and only if for every surjective module homomorphism $f : N \to M$ and every module homomorphism $g : P \to M$, there exists a homomorphism $h : P \to N$ such that $fh = g$.

A ring R is called right hereditary if the quotient of every injective right R-module is injective. We see that R is right hereditary if and only if every submodule of a projective module is projective.

It is well known a direct sum of injective modules need not be injective. However for noetherian rings we have:

Proposition 1.1.9. The following conditions on a ring R are equivalent:

1. R is right noetherian.
2. Every direct sum of injective right R-modules is injective.
3. Every countable direct sum of injective right R-modules is injective.

Definition 1.1.10. A ring R is called *a right V-ring* if each simple right R-module is injective.

Theorem 1.1.11. The following properties of a ring R are equivalent:

1. R is a right V-ring.
2. $J(M) = 0$ for every right R-module M.
3. $J(M) = 0$ for every cyclic right R-module M.
4. Every right ideal of R is an intersection of maximal right ideals.

Proof. (1) $\Longrightarrow$ (2). Let M be a right R-module and $0 \neq m \in M$. By Zorn's Lemma there is a submodule A of M maximal with respect to $m \notin A$. Let D be the intersection of all submodules properly containing A. We see that $m \in D$ and D/A is simple. So D/A is injective and $M/A = D/A \oplus K/A$, where K is a submodule of M. But $m \notin K$. Then maximality of A implies that $K = A$ and so A is maximal. Therefore $J(M) = 0$.

(2) $\Longrightarrow$ (3). Clear.

(3) $\Longrightarrow$ (4). Clear.

(4) $\Longrightarrow$ (1). Let S be a right simple R-module, and I a right ideal of R. If $0 \neq \alpha \in Hom_R(I, S)$ and $K = Ker(\alpha)$, then there is a maximal right ideal M such that $K \subseteq M$, and $I \nsubseteq M$. Since I/K is a simple R-module, then $M \cap I = K$. Therefore $R/M = (M+I)/M \cong I/(M \cap I) = I/K \cong S$. Define $f : R \longrightarrow S$ by $f(m+r) = \alpha(r)$ ($m \in M$, $r \in I$.) The map f is well defined since $M \cap I = K = Ker(\alpha)$. This means that f is an extension of α. Therefore S is injective. □

A ring R is called a right CS ring (respectively, $C2$ ring, $C3$ ring) if the module R_R satisfies $C1$ condition (respectively, $C2$ condition, $C3$ condition). For example, $\mathbb{Z}$ is CS-ring and $C3$ ring but not a $C2$ ring. The $\mathbb{Z}$-modules $\mathbb{Z}_2$ and $\mathbb{Z}_8$ satisfy the $C1$, $C2$ and $C3$ conditions, but their direct sum $N = Z_2 \oplus Z_8$ is not a CS module. Actually, for $S = \mathbb{Z}_2 \oplus 0$ and $K = \mathbb{Z}(1 + 2\mathbb{Z}, 2 + 8\mathbb{Z})$, K is contained in only two direct summands N and $S \oplus K$ and is essential in neither. Moreover, N is not a $C2$ module since the non-summand $0 \oplus \mathbb{Z}(4 + 8\mathbb{Z})$ is isomorphic to the summand $Z_2 \oplus 0$. Hence a direct sum of CS modules, or $C2$ modules, may not inherit the same property. (cf, [31])

Let M be a right R-module. If M is an indecomposable module, then M is a $C3$ module; M is a $C1$ module if and only if M is uniform; M is a $C2$ module if and only if every monomorphism $M \longrightarrow M$ is an automorphism.

Example 1.1.12. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where F is a field. Then R is a right and left $C1$ ring, but neither a left nor right $C2$ ring.

Proposition 1.1.13. If a module M satisfies $C2$ condition, then M satisfies $C3$ condition.

Lemma 1.1.14. Let A and B be submodules of M such that $A \subseteq B$. If A is closed in B and B is closed in M, then A is closed in M.

Lemma 1.1.15.

(1) Any direct summand of a CS module is a CS module.

(2) In a quasi-continuous module M, isomorphic submodules have isomorphic closures.

Lemma 1.1.16. Let M be a module such that $M/soc(M)$ is finite dimensional. If $\oplus_{i=1}^{\infty} M_i$ is an infinite direct sum of submodules of M , then there exists an integer k such that $M_i \subseteq soc(M)$ for all $i \geq k$.

Proof. Write $S = soc(M)$, $\overline{M} = M/S$ and $A_k = \oplus_{i=1}^{k} M_i$. Given $n \geq 1$, there exists a submodule $U \subseteq M_{k+1}$ such that $\overline{M}_{n+1} \cap \overline{M}_n = \overline{U}$. Write $S = (A_n \cap S) \oplus T$, so that $U \subseteq A_n + S = A_n \oplus T$ for some $T \subseteq S$. Let $\pi : A_n \oplus T \longrightarrow T$, be the projection with $ker(\pi) = A_n$. Since $U \subseteq M_{n+1}$, we have $U \cap A_n = 0$, and hence the restriction of the map π to U is monic. Thus U is semisimple, so $U \subseteq S$ and $\overline{M}_{n+1} \cap \overline{A}_n = 0$. It follows that $\overline{M}_1 \oplus \overline{M}_2 \oplus \overline{M}_3 \oplus .....$ is a direct sum; so since $\overline{M}$ is finite dimensional. there exists k such that $\overline{M}_i = 0$ for all $i \geq k$. Thus $M_i \subseteq S = soc(M)$ for all $i \geq k$, as required.

Proposition 1.1.17. Let M be a CS module. If $M/soc(M)$ is finite dimensional, then $M = K \oplus S$ where K is finite dimensional and S is semisimple.

Proof. (1). Write $S = soc(M)$ and let T be a closure of S in M. Since M is CS-module we can write $M = T \oplus K$ for some submodule K of M. Then $K \hookrightarrow M/soc(M)$; so K is finite dimensional and T is CS module by Lemma 1.1.15. So without loss of generality, we may assume that M has an essential socle. Suppose that S_1 is not a closed simple submodule of M, and let $C(S_1)$ be a closure of S_1 in M. As M is CS, set $M = C(S_1) \oplus M_1$. If S_2 is non-closed simple submodule of M_1, write $M = C(S_1) \oplus C(S_2) \oplus M_2$. If this continues indefinitely then Lemma 1.1.16 shows that some $C(S_m)$ will be in $soc(M)$; a contradiction. So, we may assume without loss of generality that every simple submodule of M is closed in M. Now suppose that D is a finitely generated submodule of M such that D has an infinitely generated socle, and let $soc(D) = \oplus_{i=1}^{\infty} A_i$, where each A_i is infinitely generated. Let $C(A_i)$ be a closure of A_i in M. Then $\oplus_{i=1}^{\infty} C(A_i)$ is an infinite direct sum of submodules of M, and by Lemma 1.1.16 there exists an integer k such that $C(A_i) = A_i$ for all $i \geq k$. By the CS hypothesis $M = A_k \oplus B_k$ for some B_k of M . Since $A_k \subseteq D \subseteq M$, it follows that $D = A_k \oplus (B_k \cap D)$, and hence A_k is finitely generated, a contradiction. Thus $soc(D)$ is finitely generated. Since every simple submodule of D is a summand, then by splitting off all the submodules of D, we can write $D = S \oplus N$, where S is semisimple and N has zero socle. Because $soc(D)$ is essential in D we infer that $N = 0$ and D is semisimple. Thus M is semisimple, as required. □

A ring R is called *semiregular* if R/J is von Neumann regular and idempotents lift modulo J; equivalently if, for each $a \in R$ there exists $e^2 = e \in aR$ such that $(1-e)\,a \in J$.

Lemma 1.1.18. Given M_R, write $S = end\,(M)$ and $\overline{S} = S/J\,(S)$, and assume that S is semiregular and $J\,(S) = \{\alpha \in S \mid ker\,(\alpha) \subseteq^{ess} M\}$. Then:

1. If $\pi^2 = \pi$ and $\tau^2 = \tau$ in S satisfy $\overline{\pi}\overline{S} \cap \overline{\tau}\overline{S} = 0$, then $\pi M \cap \tau M = 0$.

2. If M satisfies the $C3$ condition and $\sum_{i\in I} \overline{\pi}_i\overline{S}$ is a direct in $\overline{S}$ where $\pi_i^2 = \pi_i \in S$ for each i, then $\sum_{i\in I} \pi_i M$ is direct in M.

3. If M is quasi-continuous and $\sum_{i\in I} \pi_i M$ is direct in M where $\pi_i^2 = \pi_i \in S$ for each i, then $\sum_{i\in I} \overline{\pi}_i\overline{S}$ is a direct in $\overline{S}$.

Theorem 1.1.19. Let M_R be a continuous module with $S = end\,(M_R)$. Then:

1. S is semiregular and $J\,(S) = \{\alpha \in S \mid ker\,(\alpha) \subseteq^{ess} M\}$.

2. $S/J\,(S)$ is right continuous.

3. If M is quasi-injective, then $S/J\,(S)$ is right selfinjective.

Theorem 1.1.20.(*Utumi's Theorem*, cf, [53]) If R is right continuous, then R is semiregular, $Z_r = J$ and R/J is right continuous.

Proof. Take $M = R_R$ in Theorem 1.1.19. and note that

$J\,(R) = \{a \in R \mid ker\,(a\cdot) \subseteq^{ess} R\}$ where $ker\,(a\cdot) = r\,(a)$. □

1.2. Quasi-Frobenius Rings (cf, [41], [42], [43], [51], and [47])

In this section we derive some of the classical characterizations of quasi-Frobenius rings. However this requires some preliminary work.

A right ideal T is called *extensive* if every R-linear map $\alpha : T \longrightarrow R$ extends to a left multiplication that is $a\cdot : R \longrightarrow R$ with $a\cdot\,|_T = \alpha$.

Lemma 1.2.21. (cf, [4]) Let A and B be right ideals of R.

1. If $A + B$ is extensive then $l\,(A \cap B) = l\,(A) + l\,(B)$.

2. Conversely, if $l\,(A \cap B) = l\,(A) + l\,(B)$ and $\alpha : A + B \longrightarrow R$ is an R-linear map such that the restrictions $\alpha\,|_A$ and $\alpha\,|_B$ are given by left multiplication, then α is given by left multiplication.

Lemma 1.2.22.(*Ikeda-Nakayama Lemma,* cf, [56]) A ring R is called right F-injective if it satisfies any of the following equivalent statements:

1. Every R-linear map from a finitely generated right ideal to R extends to R.

2. R satisfies the following two conditions:

(a) $l(A \cap B) = l(A) + l(B)$ for all finitely generated right ideals A and B of R.

(b) $lr(a) = Ra$ for all $a \in R$.

Corollary 1.2.23. If R is right self-injective, then

1. $l(A \cap B) = l(A) + l(B)$ for all right ideals A and B of R.

2. $lr(L) = L$ for all finitely generated left ideals L of R.

Lemma 1.2.24. Assume that $lr(a) = Ra$ for all $a \in R$ and that $l(T_0 \cap T) = l(T_0) + l(T)$ for all right ideals T_0 and T of R with T_0 finitely generated. Then every R-linear map $\alpha : T \longrightarrow R$ with finitely generated image extends to $R \longrightarrow R$.

Lemma 1.2.25. A ring R is right noetherian if and only if a right R-module H exists such that every right R-module embeds in a direct sum of copies of H.

A module C is said to cogenerate a module M if M can be embedded in a direct product C^I of copies of C, and C is called a cogenerator if it cogenerates every right module. In this context the following Lemma is useful.

Lemma 1.2.26. A module C_R is a cogenerator if and only if $\cap \{ker(\lambda) \mid \lambda : M \longrightarrow C\} = 0$ for all modules M_R.

A right R-module G_R is called *a generator* if every module is an image of a direct sum $G^{(I)}$ for some set I. The cogenerators are defined dually to the generators. For injective cogenerators we have.

Proposition 1.2.27. If E_R is an injective module, then E is a cogenerator if and only if every simple right module can be embedded in E.

Proposition 1.2.28. Let $\{K_i \mid i \in I\}$ be a system of distinct representatives of the simple right R-modules, and write $C = \oplus_{i \in I} E(K_i)$. Then

1. C_R is a cogenerator.

2. C_R embeds in every cogenerator.

The module $C = \oplus_{i \in I} E(K_i)$ in Proposition 1.2.27 is called a minimal cogenerator for the category of right R-modules.

Definition 1.2.29. A ring R is called a *right Kasch ring* if every simple right module K embeds in R_R; equivalently if R_R cogenerates K.

Every semisimple artinian ring is right and left Kasch. If R is local ring, then R has only one simple right module and R is right Kasch if and only if $S_r \neq 0$.

Proposition 1.2.30. The following are equivalent for a ring R:

1. R is right Kasch.
2. $Hom\,(M, R_R) \neq 0$ for every finitely generated right R-module M.
3. $l\,(T) \neq 0$ for every proper (respectively maximal) right ideal T of R.
4. $rl\,(T) = T$ for every maximal right ideal T of R.
5. $E\,(R_R)$ is a cogenerator.

Proof. (1) $\Longrightarrow$ (2). Let M be a finitely generated right R-module. Then M has a maximal submodule N and $M/N \hookrightarrow R$. Thus $hom\,(M, R_R) \neq 0$.

(2) $\Longrightarrow$ (3). Let L be a proper right ideal and $0 \neq \sigma \in hom\,(R/T, R)$. Then $\sigma\,(1 + T) = a \neq 0$ and $a \in l\,(T)$.

(3) $\Longrightarrow$ (4). If T is a maximal right ideal, we have $T \subseteq rl\,(T) \neq R$ and this means that $T = rl\,(T)$.

(4) $\Longrightarrow$ (5). Let M be a simple right R-module. Then $M \cong R/T$ where T is a maximal right ideal of R. Let $0 \neq a \in l\,(T)$. Then $\gamma : R/T \longrightarrow R$ is well defined by $\gamma\,(r + T) = ar$. Since $T \subseteq r\,(a) \neq R$, we have $T = r\,(a)$, which shows that γ is monic. Thus $M \hookrightarrow R \subseteq E\,(R)$. Therefore $E\,(R)$ is a cogenerator by Proposition 1.2.27.

(5) $\Longrightarrow$ (1). If K_R is simple let $\sigma : K \longrightarrow E\,(R)$ be monic. Then $\sigma\,(K) \cap R \neq 0$ because $R \subseteq^{ess} E\,(R)$, so $\sigma\,(K) \subseteq R$ because $\sigma\,(K)$ is simple. □

Corollary 1.2.31. A right selfinjective ring R is right Kasch if and only if $rl\,(T) = T$ for every (maximal) right ideal T of R.

Lemma 1.2.32.(*Nakayama's Lemma*) If M_R is finitely generated, then:

1. If $MJ = M$ then $M = 0$.
2. $MJ \subseteq^{sm} M$.

Definition 1.2.33. A ring R is called *semiperfect* if R/J is semisimple and idempotents lift modulo J. R is called *semiprimary* if it is semiperfect and J is nilpotent. An idempotent e in a ring R is called a *local idempotent* if eRe is a local ring.

A right R-module M_R has a *projective cover* if there is an epimorphism $\alpha : P \longrightarrow M$ where P is projective and $ker\,(\alpha)$ is small in P.

Theorem 1.2.34. For a ring R the following conditions are equivalent:

1. R is semiperfect.

2. Every finitely generated right R-module has a projective cover. That is: if M is a finitely generated right R-module, there is an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ where P is projective, and K is small of P.

3. Every principal right R-module has a projective cover.

4. Every simple right R-module has a projective cover.

5. R is I-finite and primitive idempotents in R are local.

6. $1 = e_1 + e_2 + ... + e_n$ where the e_i are local, orthogonal idempotents.

Lemma 1.2.35. Suppose that R is a semiperfect ring in which $S_l \subseteq^{ess} R_R$. Then R is left Kasch.

Lemma 1.2.36. Assume that R is right self-injective, semiperfect ring with $S_r \subseteq^{ess} R_R$. Then R is right and left Kasch.

Definition 1.2.37. A ring R is called *left perfect* if every left R-module has a projective cover. For example, semiprimary and right artinian rings are left and right perfect. (cf, [28])

Theorem 1.2.38. The following statements are equivalent:

1. R is left perfect.

2. R has the DCC on principal right ideals.

3. R has the DCC on finitely generated right ideals.

4. Every left R-module has the ACC on cyclic submodules.

5. Every nonzero right R-module has a minimal submodule, and R has no infinite set of orthogonal idempotents.

Recall that a ring R is called a *dual ring* if $rl\,(T) = T$ for all right ideals T, and $lr\,(L) = L$ for all left ideals L. (cf, [25])

Theorem 1.2.39. The following statements are equivalent for a ring R :

1. R is quasi-Frobenius.

2. R is right or left artinian, right or left self-injective ring.

3. R is right or left noetherian, right or left self-injective ring.

4. R has ACC on right or left annihilators and R is right or left self-injective.

5. R is right or left noetherian and a dual ring.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). are obvious.

(4) $\Longrightarrow$ (5). We may assume that R is right selfinjective. Now we have two cases:

Case 1. R has ACC on left annihilators.