Ain Shams University,
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On convex bodies and local structures of Banach spaces

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Department of Mathematics,
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To all my family

To all my lecturers

To everyone advised me

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IN THE NAME OF ALLAH MOST GRACEFUL MOST MERCIFUL

"BESM ELLAH ERRAHMAN ERRAHEM"

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Introduction

Progress in the theory of linear spaces often depends on the discovery of new results in the geometry of Euclidean spaces. This due to the fact that many problems in the theory of Banach spaces may be reduced to the finite-dimensional case. In fact, one characterization of finite dimensional spaces is a classical result known as the Heine-Borel Theorem which states that " a normed linear space E is finite dimensional if and only if the unit ball of E is **compact** ".

In [1], Dvoretsky had shown the importance of **compactness** in his theorem which states that " if C is a compact set with non-void interior and symmetric about the origin in an Euclidean space of sufficiently high dimension, then there exists a k-dimensional subspace whose intersection with C is nearly spherical ",i.e., given $\epsilon > 0$, $1 > \epsilon > 0$, and a positive integer k then there exists an integer $N = N(k, \epsilon)$ such that if C is a compact set with non-void interior and symmetric about the origin in E^n, $n \geq N$, there is a subspace E^k for which $\alpha(C \cap E^k) < \epsilon$".

Where $\alpha(C) = inf\{ \epsilon > 0 :$ The convex set C is spherical to within $\epsilon\}$; a convex set C in a linear normed space is called *spherical to within* ϵ, $1 > \epsilon > 0$, if there exists in the flat space generated by C two concentric balls $B_1$and B_2 of radii $r(1-\epsilon)$ and r such that $B_1 \subset C \subset B_2$.

In this project, we study many subjects one of them in chapter one various types of compactness in some normed and metric spaces. For, it is well known that "compactness" is central in mathematical analysis which is of paramount importance from the point of view of numerous applications.

In chapter two, we study and discuss Gâteaux differentiation and Fréchet differentiation and their properties since both derivatives are often used to formalize the functional derivative commonly used in Physics, particularly Quantum field theory and also one of the main tools of nonlinear functional analysis is the " linearization " of maps and a natural way to do so is to use derivatives.

Also we study Lipschitz function and its properties since the study of uniformly continuous and, in particular, Lipschitz functions between Banach spaces are important. Such functions appear in many contexts, and their study involves a rich interplay between topology, geometry of Banach spaces, geometric measure theory, classical analysis, probability and even combinatory. The study of uniformly continuous and Lipschitz maps is naturally connected to the classification of Banach spaces and their subsets (especially balls) with respect to uniform homeomorphisms and bi-Lipschitz maps. Also Lipschitz functions is the entrance to Lipschitz- free Banach spaces and the importance of this type of Banach spaces is to investigate the following general problem: if X and Y are Lipschitz isomorphic Banach spaces, that is, if there exist a bijective and bi-Lipschitz map $F: X \to Y$,are X and Y linearly isomorphic?

Free spaces are naturally relevant to this problem. It is known that the answer to this question is negative in full generality, but it remains an important open problem in the separable case. The map δ defines a nonlinear isometric embedding from X into $F(X)$, with a linear left inverse. Finally, moving to free spaces allows linearization of Lipschitz maps and this opens the way for applications of the linear theory to nonlinear problems. The last section of chapter two is about some results on convex bodies and the importance of studying convex bodies is indicated in the previous.

In chapter three, we study and discuss the modulus of continuity of functions and its properties and how we can identify the uniformly continuous functions by their modulus of continuity. Also we discuss and study algebra of modulus of continuity which is very useful for our new results.

Also in chapter three we clear the importance and properties of the space of bounded real valued functions on normed spaces or on metric spaces.

Finally, we get new results beginning from that In [1] it was proved that any uniformly continuous function defined on a subset of a metric space X with values in $\boldsymbol{l}_{\infty}(\Omega)$, with modulus of continuity dominated by a nondecreasing subadditive function $w(t)$ satisfying $lim_{t\to 0} w(t) = 0$, can be extended to a uniformly continuous function F on the whole space whose modulus of continuity $w_F(t) \leq w(t)$. Our new result is to give a general way to construct a common uniformly continuous extension for a finite collection of uniformly continuous functions with pair wise disjoint domains and with modulus of continuity dominated by a certain subadditive function.

Summary

This thesis consists of three chapters

In chapter 1, we write the preliminaries, the fundamental concepts and theorems in functional analysis that we will use through the thesis. Also we study the compactness of linear operators.

In chapter 2, we study Gâteaux differentiation and Fréchet differentiation and their properties. Then Discuss and study Lipschitz functions, Lipschitz retraction, absolute Lipschitz retract and Lipschitz free Banach spaces. Finally we discuss some results on convex bodies.

In chapter 3, we study the modulus of continuity and its properties then we discuss algebra of it. Also we study the space of bounded real valued functions on normed spaces or on metric spaces and its properties. Finally we get new results.

Chapter One

Preliminary Concepts

This chapter presents the concepts and propositions that will be used in the next chapters. Besides some basic definitions and facts from functional analysis. The purpose of this chapter is to give the reader a survey of the material that will be used in the later chapters.

1.1 Notations

- $\mathbb{R}$ is the field of real numbers.
- Z is the set of integers.
- N is the set of natural numbers.
- for two Banach Spaces X and Y :

We denote by $L(X, Y)$ the Banach Space of all continuous (bounded) linear operators from X into Y endowed with the usual norm.

$$\|f\| = sup\{ \|f(x)\| : \ x \in X, \ \|x\| \leq 1\}.$$

- For every operator $f \in L(X, Y)$

$D(f)$ denotes the domain of f.

$R(f)$ Denotes the range of f.

- If X is a Banach Space :

We shall denote by $X^* = L(X, \mathbb{R})$ its dual (conjugate) space. The identity operator of X is denoted by I_X, and $U_X$the unit ball of X " $U_X = \{x \in X : \|x\| \leq 1\}$ ".We may simply write I and U.

- $l_\infty(X)$ denotes the space of all bounded real-valued functions of X

$$l_\infty(X) = \{ f \quad ; \quad f: X \rightarrow \mathbb{R}, \quad sup_{x\in X} |f(x)| < \infty\}$$

with the norm

$$\|f\| = sup_{x\in X} |f(x)|$$

- $C_{[o,1]}$ denotes the space of all continuous functions f defined on the interval [0,1] with the norm

$$\|f\| = \{ |f(x)| : x \in [0,1] \}.$$

1.2 Basic Definitions

In this section we list some well-known definitions that will be used later on.

Definition 1.2.1 (Conjugate operator) [31]

The conjugate (adjoint) operator of an operator $f \in L(X, Y)$ is the mapping

$$f^* : Y^* \rightarrow X^*$$

defined by

$$f^*\big(g(x)\big) = g\big(f(x)\big) \quad \forall\, x \in X\,, \ \forall\, g \in Y^*$$

"or in inner product spaces $\langle x, f^* g\rangle = \langle fx, g\rangle$". Of course $f^* \in L(Y^*, X^*)$ and $\|f^*\| = \|f\|$.

Definition 1.2.2 (Isometry, isometric spaces) [9]

Let $E = (E, d_1)$and $F = (F, d_2)$ be metric spaces. A mapping f of E into F is said to be isometric or an isometry if f preserves distance, that is, if for all $x, y \in E$

$$d_2(f(x), f(y) = d_1(x, y)$$

Two metric spaces E and F are called isometric spaces if there is a bijective (one to one and onto) isometry of E onto F.

Definition 1.2.3 (Isomorphism) [8]

An isomorphism of a normed space E into a normed space F is a bijective linear continuous operator $f : E \to F$ whose inverse is continuous. E and F are called isomorphic normed spaces if there is an isomorphism of E onto F. In other words, E and F are isomorphic if and only if there exist a linear operator f with $D(f) = E$, $R(f) = F$ and there exist positive constants a, b such that

$$a\|x\| \leq \|f(x)\| \leq b\|x\| \quad \forall x \in E.$$

Definition 1.2.4 (Base at the point) [28]

Let E be a topological space. A collection $\mathcal{B}$ of open sets of E is a base at the point $x \in E$ if for any neighborhood N of x, there exists an open set $O \in$ B such that $x \in O \subseteq N$.

Let X be a Banach space and let X^* be its conjugate space, the topologies considered here are the strong topology, the weak topology and, when the Banach space is the conjugate space the $weak^*$ topology.

Definition 1.2.5 (Strong Topology) [10]

The metric topology induced by the norm of X is called the norm or strong topology for X. In other words, the strong topology for X is defined by the neighborhoods of the points $x \in X$ as follows:

$$N(x, \varepsilon) = \{y : \|y - x\| < \varepsilon\}$$

Thus,"$\{x_n\}$ converges to x in the norm topology" means that

$$\|x_n - x\| \to 0 \text{ as } n \to \infty$$

and the sequence$\{x_n\}$is called strongly convergent to x.

Definition 1.2.6 (Weak Topology) [10]

The weak topology for X is obtained by taking as a base at the points x of X the neighborhoods

$$N(x, M, \epsilon) = \{\, y : \ |f(y - x)| < \epsilon\,,\ f \in M\,\},$$

where $\epsilon > 0$ and M is a finite subset of X^*. Thus, "$\{x_n\}$ converges to x in the weak topology" means that

$$|f(x_n) - f(x)| \to 0 \text{ as } n \to \infty \quad \forall f \in X^*$$

and the sequence $\{x_n\}$ is called weakly convergent to x.

Definition1.2.7 (The weak Topology) [5]

Let X be a Banach space. For each $f \in X^*$, we associate a map

$$\varphi_f : X \to \mathbb{R}$$

defined by

$$\varphi_f(x) = f(x) \quad \forall\, x \in X.$$

As f ranges over X^*, we obtain a family $\{\varphi_f\}_{f \in X^*}$ of maps from X into $\mathbb{R}$. The weak topology on X (denoted by ω) is the smallest topology on X (because it has the minimum open sets) which makes the maps φ_f continuous.

Definition 1.2.8 ($\boldsymbol{Weak}^*$ Topology) [28]

The $weak^*$ topology for the conjugate space X^* can be introduced by taking as a base at the points f of X^* the neighborhoods

$$N(f, M, \varepsilon) = \{\, g : \quad |(g - f)(x)| < \epsilon\,, \quad x \in M\},$$

where $\epsilon > 0$ and M is a finite subset of X. Thus, $\{f_n\}$ converges to f in the $weak^*$ topology means that

$$|\, f_n(x) - f(x)| \to 0 \text{ as } n \to \infty \quad \forall\, x \in X$$

and the sequence $\{f_n\}$ is called $weak^*$ convergent to f.

Definition 1.2.9 ($\boldsymbol{Weak}^*$ Topology) [5]

For each $x \in X$ we consider the map

$$\varphi_x : X^* \to \mathbb{R}$$

defined by

$$\varphi_x(f) = f(x).$$

As x ranges over X, we obtain a family $\{\varphi_x\}_{x \in X}$ of maps