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On MS-algebras and double MS-algebras

A thesis Submitted

By

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Contents

Preface

In 1890 Richard Dedekind was working on a revised and enlarged edition of Dirichlet's Vorlesungen ber Zahlentheorie (Lectures in the theory of numbers), and asked himself the following question: Given three subgroups A, B, C of an abelian group G, how many different subgroups can you get by taking intersections and sums, e.g., $A+B$, $(A+B)\cap C$, etc. In looking at this and related questions, Dedekind was led to develop the basic theory of lattices, which he called "Dualgruppen". In fact, Dedekind was ahead of his time in making the connection between modern algebra and lattice theory, and so nothing much happened in lattice theory for the next thirty years. Then, with the development of universal algebra in the thirties of the last century, Garrett Birkhoff started the general development of lattice theory. Birkhoff himself, Valèere Glivenko, Karl Menger, John von Neumann, Oystein Ore, and others had developed enough of this new field for Birkhoff to attempt to sell it to the general mathematical community, which he did with astonishing success in the first edition of "Lattice Theory", [12]. Nowadays, the tentacles of lattice theory extend into algebra, analysis, topology, logic, combinatorics, linear algebra, geometry, category theory, probability and computer science, see [27], [31], [32], and [39].

De Morgan Stone algebras (or simply MS-algebras) have been first introduced by T.S. Blyth and J.C. Varlet as a common abstraction of de Morgan algebras and Stone algebras ([15], [17]) . The triple construction of MS-algebras by means of Kleene algebras and distributive lattices has been established and investigated by Blyth and Varlet in [14]. Moreover, a very interesting quadruple representation of the MS-algebras has been proved to be a very satisfactory tool in characterizing and analyzing the structure of MS-algebras, [16]. T. Katrinak added new ideas to the theory of MS-algebras via introducing and characterizing the class of modular double S-algebras, [37]. In 1988, a marvelous comparison between triples and quadruples has been established by T. Katrinak and K. Mikula in [38]. Continuing this clue; characterizations of ideals, filters, fixed points and congruences of MS-algebras have been deduced, see [19], [20], [29], and [43]). An interesting analysis of generalized MS-algebras which are non-distributive MS-algebras with has been handled in [2], [3], [42], and [44].

The class of double MS-algebras and the class of regular double MS-algebras have been proved to be a very interesting and convenient subclasses of the class of MS-algebras, [4], [18], [26].

A. Badawy, D. Guffova and M. Haviar, [6], established an important characterization of decomposable MS-algebras in terms of decomposable MS-triples. Moreover, they deduced a one-to-one correspondence between decomposable MS-algebras and decomposable MS-triples. Congruences, homomorphisms, subalgebras, and filters of decomposable MS-algebras have been studied intensively and extensively in [5] and [8].

In this thesis we try to dig deeper in the theory of MS-algebras and double MS-algebras. The main objective of this thesis is threefold. First, we aim to investigate completeness properties of decomposable MS-algebras. Second, we aim to construct and characterize decomposable double MS-algebras. Third, we aim to study direct products and ideals of decomposable MS-algebras.

This thesis consists of four chapters which are organized as follows:

In the first chapter we assemble the preliminaries and basic material to be used in the thesis. We provide a brief survey of the basic definitions and results concerning Lattices, MS-algebras, decomposable MS-algebras and double MS-algebras. For a sake of completeness, some important constructive proofs are included.

The second chapter consists of three sections. In the first section we introduce and investigate the notions of complete decomposable MS-algebras and complete decomposable MS-triples. our main result of this section is that a decomposable MS-algebra L constructed from the decomposable MS-triple (M, D, φ) is complete if and only if the triple (M, D, φ) is complete. In the second section we introduce complete triple homomorphisms of complete decomposable MS-algebras. Actually, we provide a characterization of complete homomorphisms of complete decomposable MS-algebras in terms of complete triple homomorphisms. The third section is devoted to study some fill-in problems concerning decomposable MS-triples. Roughly speaking, given a complete de Morgan algebra M and a conditionally complete distributive lattice D, a fill-in problem is concerned with constructing a homomorphism φ so that (M, D, φ) is a complete decomposable MS-triple.

The third chapter is divided into four sections. In the first section we introduce and study the notion of decomposable double MS-algebras. We obtain necessary and sufficient conditions for a decomposable MS-algebra to become a decomposable double MS-algebra. In the second section, we construct decomposable double MS-algebras from decomposable MS-quadruples as a generalization of the construction of decomposable MS-algebras by means of decomposable MS-triples. We show that there exists a one-to-one correspondence between decomposable double MS-algebras and decomposable MS-quadruples. We construct decomposable double K_2-algebras using decomposable K_2-quadruples and double Stone algebras using Stone quadruples. In the third section we confine our attention to the study of subalgebras of decomposable double MS-algebras. One of the main results in this section is the characterization of the greatest Stone subalgebra of a decomposable double MS-algebras. The fourth section is devoted to investigate isomorphisms of decomposable double MS-algebras. We prove that two decomposable double MS-algebras are isomorphic if and only if the associated MS-quadruples are isomorphic.

The fourth chapter consists of three sections. In the first section we prove some results on subalgebras of the direct product of decomposable MS-algebras. In the second section we investigate homomorphic image and inverse homomorphic image of subalgebras of decomposable MS-algebras. One of the main results is the proof of a universal mapping property for direct products of decomposable MS-algebras. In the third section we introduce and investigate the notion of MS-ideals of MS-algebras. We study the relation between MS-ideals and some other known ideal of MS-algebras. We round off by deducing the influence of homomorphisms on MS-ideals as well as the relation between congruences and MS-ideals.

The main results extracted from this thesis are included in the following publications:
(1) Ahmed Gaber, Abdel Mohsen Badawy and Salah El-din S.Hussein, On decomposable MS-algebras, accepted for publication in Italian Journal of Pure and Applied Mathematics.
(2) Abdel Mohsen Badawy, Essam El-seidy and Ahmed Gaber, MS-ideals of MS-algebras, Applied Mathematical Sciences, Vol. 13, 2019, no. 7, 347 - 357 .
(3) Abdel Mohsen Badawy and Ahmed Gaber, Complete decomposable MS-algebras, accepted for publication in Journal of the Egyptian Mathematical Society.
(4) Abdel Mohsen Badawy, Salah El-din S.Hussein and Ahmed Gaber, Quadruple constructions of decomposable double MS-algebras, submitted for publication.

Chapter 1

Preliminaries

In this chapter we introduce the background material which we need in this thesis. However, we just provide a brief survey of the basic definitions and elementary results concerning lattices, MS-algebras, decomposable MS-algebras and double MS-algebras. For a sake of completeness, some important constructive proofs are included. For details on lattices we refer to [9], [11], [23], and [33] ; for details on MS-algebras and decomposable MS-algebras we refer to [6], [15], [21], [22], and [46]; for details on ideals and filters of MS-algebras we refer to [1], [7], and [40]; and for details on double MS-algebras we refer to [13], [18], and [34].

1.1 Lattices and distributive lattices

Definition 1.1.1 ***A lattice*** *is an algebra* $(L, \wedge, \vee)$ *satisfying, for all* $x, y, z \in L$,

(1) $x \wedge x = x$ *and* $x \vee x = x$,

(2) $x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$,

(3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ *and* $x \vee (y \vee z) = (x \vee y) \vee z$,

(4) $x \wedge (x \vee y) = x$ *and* $x \vee (x \wedge y) = x$.

The fourth pair of axioms, called the **absorption law**, play an important role in the proof of the following theorem. This theorem reveals the equivalence between the notion of a lattice as an algebraic structure and the notion of a lattice as a partially ordered set.

Theorem 1.1.2 *In a lattice L, define* $x \leqslant y$ *if and only if* $x \wedge y = x$. *Then* $(L, \leqslant)$ *is an ordered set in which every pair of elements has a greatest lower bound (infimum) and a least upper bound(supremum). Conversely, given an ordered set L with that property, define* $x \wedge y = inf(x, y)$ *and* $x \vee y = sup(x, y)$. *Then* $(L, \wedge, \vee)$ *is a lattice.*

In light of the Theorem 1.1.2, we see that, for any $a, b \in L$, $a \leqslant b$ if and only if $a \vee b = b$. Equivalently, $a \leqslant b$ if and only if $a \wedge b = a$. Moreover, Theorem 1.1.2 yields the following definition of a lattice.

Definition 1.1.3 *A lattice is a partially ordered set* $(L, \leqslant)$ *such that* $inf\{a, b\}$ *and* $sup\{a, b\}$ *exist for any* $a, b \in L$.

Definition 1.1.4 *A lattice L is bounded if it has both 1 (the greatest element) and 0 (the least element).*

Example 1.1.5

(1) The powerset of a set forms a lattice, with inclusion being the partial order. Join and meet are union and intersection, respectively.

(2) The finite subsets of a set form a lattice, with inclusion being the partial order. Join and meet are union and intersection, respectively.

(3) The partitions of a set form a lattice, where $a \geqslant b$ *iff a is a refinement of b.*

(4) The subgroups of a group form a lattice, with inclusion being the partial order. The join of two subgroups is the subgroup generated by the two subgroups, and the meet of two subgroups is their intersection.

(5) The normal subgroups of a group form a lattice, with inclusion being the partial order. The join of two normal subgroups is the product of the two normal subgroups, and the meet of two normal subgroups is their intersection.

(6) The subrings of a ring form a lattice, with inclusion being the partial order. The join of two subrings is the subring generated by the two subrings, and the meet of two subrings is their intersection.

(7) The ideals of a ring form a lattice, with inclusion being the partial order. The join of two ideals is their sum, and the meet of two ideals is their intersection.

(8) The open subsets of a topological space form a lattice, with inclusion being the partial order. Join and meet are union and intersection, respectively.

(9) The closed subsets of a topological space form a lattice, with inclusion being the partial order. Join and meet are union and intersection, respectively.

(10) Integers form a lattice, with its usual ordering. In fact, any set with a total order is a lattice.

(11) Positive integers form a lattice, where $a \geqslant b$ iff a is a multiple of b. Join and meet are the least common multiple and greatest common divisor, respectively.

(12) Ordered pairs of integers form a lattice, where $(a, b) \geqslant (c, d)$ iff $a \geqslant c$ and $b \geqslant d$. We have $(a, b) \vee (c, d) = (max\{a, c\}, max\{b, d\})$, $(a, b) \wedge (c, d) = (min\{a, c\}, min\{b, d\})$.

(13) Any finite poset is a lattice iff it has a maximum and a minimum.

(14) The sublattices of a lattice together with the empty set form a lattice, with inclusion being the partial order. The join of two sublattices is the sublattice generated by the two sublattices, and meet of two sublattices is their intersection.

Definition 1.1.6 *A lattice L is called* ***complete*** *if $\inf_L H$ and $\sup_L H$ exist for each $\phi \neq H \subseteq L$.*

Definition 1.1.7 *A lattice L is called* ***conditionally complete*** *if every upper bounded subset of L has a supermum in L and every lower bounded subset of L has an infimum in L.*

Note that any complete a lattice is a conditionally complete lattice.

Example 1.1.8

(1) The powerset of a set is a complete lattice

(2) The subgroups of a group form a complete lattice

(3) The closed subsets of a topological space form a complete lattice

Definition 1.1.9 *Let L and L_1 be lattices. Let $f : L \to L_1$ be a mapping and let a, b be any two elements of L. Then,*

(1) f is a $\vee$-homomorphism if $(a \vee b)f = (a)f \vee (b)f$.

(2) f is a $\wedge$-homomorphism if $(a \wedge b)f = (a)f \wedge (b)f$.

(3) f is a lattice homomorphism if it is both $\vee$-homomorphism and $\wedge$-homomorphism.

(4) A lattice monomorphism or a lattice embedding is an injective lattice homomorphism.

(5) A lattice epimorphism is a surjective lattice homomorphism.

(6) A lattice endomorphism is a lattice homomorphism from a lattice L into itself.

(7) A lattice isomorphism is a bijective lattice homomorphism.

Definition 1.1.10 *A lattice homomorphism $h : L \to L_1$ of a complete lattice L into a complete lattice L_1 is called **complete** if*

$$(\inf_L H)h = \inf_{L_1} Hh \text{ and } (\sup_L H)h = \sup_{L_1} Hh \text{ for each } \phi \neq H \subseteq L.$$

Definition 1.1.11 *Let L and L_1 be lattices. A lattice homomorphism $f : L \to L_1$ is called a $(0,1)$**-homomorphism** if $(0)f = 0$ and $(1)f = 1$.*

Definition 1.1.12 *Two lattices L and L_1 are isomorphic (written $L \simeq L_1$) if there is an isomorphism between them .*

Remark 1.1.13 *if Φ is a statement about a lattice, and we replace all occurrences of $\leqslant$ by $\geqslant$ (or $\wedge$ by $\vee$), and vice versa, we get the dual statement of Φ . This is known as The duality principle and is based on the simple observation that the definition of a lattice is self-dual. That is, if L is a lattice, then its dual L_d is also a lattice.*

Definition 1.1.14 *A **sublattice** S of a lattice L is a non empty subset of L, such that for every pair of elements $a, b \in S$ both $a \vee b$ and $a \wedge b$ are in S where $\vee$ and $\wedge$ are the lattice operations of L*

Definition 1.1.15 *A sublattice S of L is called a **bounded sublattice** if it has both 1 (the greatest element) and 0 (the least element) of L.*

Definition 1.1.16 *Let L be a lattice. Then,*

(1) an ideal I of L is a nonempty subset of L such that,

(i) $a, b \in I \Rightarrow a \vee b \in I$*,*

(ii) $a \in I, x \leqslant a \Rightarrow x \in I$ *for all* $x \in L$*.*

(2) Dually, a filter F of L is a nonempty subset of L such that,

(i) $a, b \in F \Rightarrow a \wedge b \in F$*,*

(ii) $a \in F, x \geqslant a \Rightarrow x \in F$ *for all* $x \in L$*.*

The set of all ideals of L is denoted by $I(L)$ and the set of all filters of L is denoted by $F(L)$.

Definition 1.1.17 *Let L be a lattice. Then,*

(1) An ideal I of L is the ***principal ideal*** *generated by an element* $a \in L$*, written* $I = (a]$*, if* $I = \{x \in L : x \leqslant a\}$*.*

(2) Dually, a filter F of L is the ***principal filter*** *generated by an element* $a \in L$*, written* $F = [a)$*, if* $F = \{x \in L : x \geqslant a\}$*.*

(3) A proper ideal I of L is ***maximal*** *if for any ideal J of L,*

$$I \subseteq J \subseteq L \Rightarrow J = I \quad or \quad J = L.$$

(4) Dually, a proper filter F in L is ***maximal*** *if for any filter G of L,*

$$F \subseteq G \subseteq L \Rightarrow G = F \quad or \quad G = L.$$

(5) A proper ideal I of L is ***prime*** *if for any* $a, b \in L$*,*

$$a \wedge b \in I \Rightarrow a \in I \quad or \quad b \in I.$$

(6) Dually, A proper filter F of L is ***prime*** *if for any* $a, b \in L$*,*

$$a \vee b \in I \Rightarrow a \in F \quad or \quad b \in F.$$

Theorem 1.1.18 *The following two identities are equivalent in any lattice L for all $x, y, z \in L$,*

$$\begin{aligned} x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\ x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z). \end{aligned}$$

Definition 1.1.19 *A lattice L is* ***distributive*** *if for all $x, y, z \in L$,*

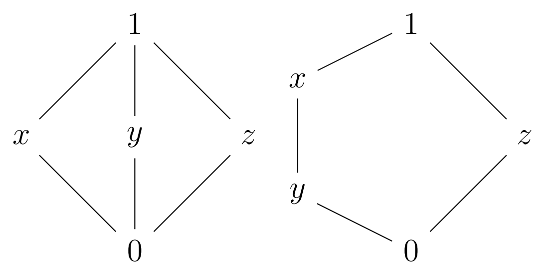
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Example 1.1.20

(1) The lattice of subsets of a set X is a distributive lattice since $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, for all $A, B, C \subseteq X$.

(2) Consider the lattice of subgroups of the Klein four group defined by $V_4 = \langle a, b : a^2 = b^2 = (ab)^2 = e \rangle$. Let $c = ab$. Then $\langle a \rangle \wedge (\langle b \rangle \vee \langle c \rangle) = \langle a \rangle \wedge V_4 = \langle a \rangle$, while $(\langle a \rangle \wedge \langle b \rangle) \vee (\langle a \rangle \wedge \langle c \rangle) = \langle e \rangle \vee \langle e \rangle = \langle e \rangle$, the trivial subgroup. So, the lattice is not distributive.

Theorem 1.1.21 *[33] A lattice L is distributive if L does not contain a sublattice of either of the form M_3(diamond) or of the form N_5(pentagon)*



M_3 N_5

Theorem 1.1.22 *Let L be a lattice. Then*

(1) L is distributive if and only if $I(L)$ is distributive.