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Candidate :- Ahmed Ibrahim Mohamed Mostafa Marie

Approved by Advisors

Dr . Samy Mohamed Mostafa
Assistant Professor of Pure Mathematics
Mathematics Department
Faculty of Educaton
Ain Shams University

Dr . Ragab Abd El Kader Omar
Lecturer of Pure Mathematics
Mathematics Department
Faculty of Education
Ain Shams University

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SOME PROPERTIES OF FUZZY IDEALS IN BCI- AlGEBRAS

A THESIS

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To

Department of Mathematics
Faculty of Education , Ain Shams University

By

Ahmed Ibrahim Mohamed Mostafa Marie

Faculty of Education , Ain Shams University

Under Supervisions

Dr. Samy Mohamed Mostafa
Ass. Prof. of Pure Mathematics
Mathematics Department
Faculty of Educaton
Ain Shams University

Dr. Ragab Abd El Kader Omar
Lecturer of Pure Mathematics
Mathematics Department
Faculty of Education
Ain Shams University

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CONTENTS

Introduction

The notion of BCK-algebras was proposed by Iami and Iseki in 1966. In the same year, Iseki introduced the notion of a BCI-algebra which is a generalization of BCK-algebra . Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK / BCI-algebras and their relationship with other structures including lattices and Boolean algebras. There is a great deal of literature has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras [5]. For the general development of BCK/BCI-algebras, the ideal theory plays an important role. The concept of fuzzy sets was first introduced by Zadeh [16] . From that time, the theory of fuzzy sets has been developed in many directions and found applications in a wide variety of fields . It application to various mathematical contexts has given rise to what is now commonly called fuzzy mathematics. Fuzzy algebras is an important branch of fuzzy mathematics, many researchers [1, 3, 27, 31] have studied some algebraic structures such as fuzzy semi-groups, fuzzy groups, fuzzy rings, fuzzy ideals, fuzzy modules, fuzzy vectors spaces, fuzzy category, fuzzy functors and so on. In 1991, Xi [20] applied the concept of fuzzy sets to BCI, BCK and MV-algebras . In [44], Jun and Meng considered the fuzzification of p−ideals in BCI-algebras. In [40], Liu and Meng introduced the notion of fuzzy positive implicative and investigate some of their properties. Liu and Meng [39] introduced the notion of sub-implicative ideals in BCI-algebras. Also Jun [37] introduced the notion of fuzzy sub-implicative ideals of BCI-algebras and obtained some related interesting properties of these concepts. Jun [32] defined a doubt fuzzy sub-algebra, doubt fuzzy ideal, doubt fuzzy implicative ideal, and doubt fuzzy prime ideal in BCI-algebras . He got some results about it.The purpose of this thesis is to study and give some characterizations of anti fuzzy sub-implicative ideal and doubt fuzzy sub-commutative ideal . All of our new results are included in chapters II – III .

This Thesis consists of three chapters.

In the first chapter, we have given an exhaustive of the basic definitions in BCI(BCK)-algebras and fuzzy sets which are needed in the subsequent chapters and further, the history of the problem.

In the second chapter, several researchers investigated further properties of fuzzy BCI-algebras and fuzzy ideal [see {[11] , [21] , [24], [37]}]. Jun [34] gave some properties of a fuzzy commutative ideals in BCK-algebra .Liu and Meng [39] introduced the notion of sub-implicative ideal and sub-commutative ideal in BCI - algebra and investigated the properties of this ideals. Jun [32] defined a doubt fuzzy sub algebra, doubt fuzzy ideal, doubt fuzzy implicative ideal, and doubt fuzzy prime ideal in BCI- algebras. He got some results about it. Modifying this idea, in this chapter, we introduce the concept of doubt fuzzy sub-commutative (DFSC) ideal of BCI-algebra and investigate some related properties. We show that in a commutative BCI-algebras, a fuzzy subset is a doubt fuzzy ideal if and only if it is doubt fuzzy sub-commutative ideal, and a fuzzy subset of a BCI-algebra is a fuzzy sub-commutative ideal if and only if the complement of this fuzzy subset is a doubt fuzzy sub-commutative ideal. Moreover, we discuss the pre-image (image) of a doubt fuzzy sub-commutative ideal. Finally, we introduce the notion of cartesian product of doubt fuzzy sub-commutative ideal and then we characterize doubt fuzzy sub-commutative ideal of it.

In Chapter three, we refer to Jun and Meng investigated further properties of fuzzy BCI-algebras and fuzzy ideal [see {[30] , [31] , [34] , [39]}]. S .M .Mostafa [27] gave some properties of a fuzzy implicative ideal in BCK-algebra . [23] Biswas introduced the concept of anti-fuzzy subgroup. Modifying this idea, in this chapter, we introduce the concept of anti-fuzzy sub implicative (AFSI) ideal of BCI-algebra and investigate some related properties. We show that in an implicative BCI-algebras, a fuzzy subset is an anti-fuzzy ideal if and only if it is anti-fuzzy sub-implicative ideal, and a fuzzy subset of a BCI-algebra is a fuzzy sub-implicative ideal if and only if the complement of this fuzzy

subset is an anti-fuzzy sub implicative ideal. Moreover, we discuss the image (pre-image) of anti-fuzzy sub-implicative ideal. Finally, we introduce the notion of cartesian product of anti fuzzy sub implicative ideal and then we characterize anti fuzzy sub implicative ideal by it .

CHAPTER I

Preliminaries

The aim of this chapter is to give some definitions, notations, and some results of BCI-algebras which will be used in the later chapters.

1.1 Notions of BCI/BCK-algebras :

In this introductory chapter we have stated the various definitions of the terms and functions related to our course of investigation. Keeping in view just to avoid the repetitions and standard results. This Chapter is fully interwoven with all these terminologies and definitions. The study of BCI(BCK)-algebras was initiated by K.Iseki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. In particular, emphasis seems to have been put on the ideal theory of BCI/BCK-algebras.

Definition 1.1.1 ([12]):

Let X be a non-empty set with a binary operation $*$ and a constant 0. Then (X ; $*$, 0), of type (2,0), is called a BCI-algebra if it satisfies the following conditions :

(1) $((x * y) * (x * z)) * (z * y) = 0$,

(2) $(x * (x * y)) * y = 0$,

(3) $x * x = 0$,

(4) $x * y = y * x = 0$ *implies* $x = y$. For all $x, y, z \in X$.

In a BCI-algebra X , we can define a partially ordered relation $\leq$ by letting $x \leq y$ if and only if $x * y = 0$.

Equivalently (X ; $*$, 0) is a BCI-algebra if and only if the following are satisfied :

(1\) $(x * y) * (x * z) \leq (z * y)$,

(2\) $x * (x * y) \leq y$,

(3\) $x \leq x$,

(4\) $x \leq y$ *and* $y \leq x$ *implies* $x = y$.

For all $x , y , z \in X$.

In This case (X ; $*$, 0 , $\leq$) is called an Partially Ordered BCI-algebra .

Example 1.1.2 ([38]):

Let X = {0, 1, 2, 3} be a set with a binary operation $*$ given by the table :

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

It is easy to check that (X ; $*$, 0) is a BCI-algebra .

Theorem 1.1.3 ([22]):

A BCI-algebra X satisfying $(x*y)*z = x*(y*z)$ is a group in which $x * x = 0$ for every $x,y,z \in X$, where 0 is unit.

The converse of this theorem is also true ([13]) .

Theorem 1.1.4 ([12]):

In a BCI-algebra (X ; $*$, 0), the following properties hold :

(1) $(x*y)*z=(x*z)*y$ for all x, y and z $\in$ X ,

(2) $x*(x*(x*y))=x*y$,

(3) $(x*z)*(y*z)\le x*y$,

(4) $0*(x*y)=(0*x)*(0*y)$,

(5) $x*0=x$,

(6) $x\le y$ *implies* $x*z\le y*z$ *and* $z*y\le z*x$.

Theorem 1.1.5 ([22]):

A BCI-algebra X satisfying $0*x=x$ is a group in which $x*x=0$ for every $x\in X$, where 0 is the unit.

Definition 1.1.6 ([12]):

Let (X ; $*$, 0) be a BCI-algebra and let M be a nonempty subset of X . Then M is said to be a subalgebra of X if for all x , y $\in$M ; $x*y$ $\in$ M .

Example 1.1.7 ([12]):

In example (1.1.2) , it is easy to check that M = { 0 , 1 , 2 } is a subalgebra of X .

Theorem 1.1.8 ([12]):

Let X be a BCI-algebra and let M be a subalgebra of X , then

a) 0 $\in$ M ,

b) (M ; $*$, 0) is also BCI-algebra ,

c) X is a subalgebra of X ,

d) {0} is a subalgebra of X .

Definition 1.1.9 ([15]):

A BCI-algebra is said to be a BCK-algebra if it satisfies:

$0 * x = 0 \quad for\ all\ x \in X$.

Example 1.1.10 ([43]):

Let X = { 0 , 1 , 2 , 3 , 4 } in which $*$ is given by the table :

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	1	1	0	0
4	4	3	2	1	0

It is easy to check that $(X ; *, 0)$ is a BCK-algebra . In example (1.1.2) , $(X ; *, 0)$ is a BCI-algebra but not BCK-algebra because $0 * 3 = 3 \neq 0$.

Definition 1.1.11 ([9]):

A BCI-algebra X is called commutative if it satisfies:

$x \leq y \ implies \ x = y*(y* x)$.

Example 1.1.12 ([9]):

Let X = {0, a, b, c, d} be a BCI-algebra given by the table :

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	0	0
c	c	c	b	0	b
d	d	b	a	a	0

By routine calculations (X ; $*$, 0) is a commutative BCI- algebra .

In Example (1.1.10) (X ; $*$, 0) is a BCI-algebra but not commutative because $2 \leq 3$, but $3*(3*2) = 3*1=1\neq 2$.

Theorem 1.1.13 ([43]):

An algebra (X ; $*$,0) is commutative BCK-algebra if and only if it satisfies the following:

(a) $x*(x*y) = y*(y*x)$,

(b) $(x*y)*z = (x*z)*y$,

(c) $x*x = 0$,

(d) $x*0 = x$.

Theorem 1.1.14 ([9])**:**

A BCI-algebra is commutative if and only if it satisfies $x*(x*y) = y*(y*(x*(x*y)))$, for all $x, y \in$ X.

Theorem 1.1.15 ([9]):

For a BCI-algebra (X ; $*$, 0) the following are equivalent :

(a) X is commutative ,

(b) $x*(x*y) \leq y*(y*x)$,

(c) $(x*(x*y))*(y*(y*x)) =0$.

Definition 1.1.16 ([32]):

For any x , y in X denote $x \wedge y = y*(y*x)$, *where* $x \wedge y$ is a lower bound of x and y . So we can say that a BCI-algebra is commutative if it satisfies: $x \wedge y = y \wedge x$.

Definition 1.1.17 ([43]):

A partially ordered set (X , $\leq$) is said to be a lower semilattice if every pair of elements in X has a greatest lower bound " meet " $\wedge$ " .

Definition 1.1.18 ([43]):

A partially ordered set (X , $\leq$) is said to be an upper semilattice if every pair of elements in X has a least upper bound " join " $\vee$ " .

If (X , $\leq$) is both an upper and lower semilattice then it is called lattice .

Theorem 1.1.19 ([43]):

A BCK-algebra (X ; $*$, 0) is commutative if and only if it is a lower semilattice with respect to BCK-order $\leq$.

Theorem 1.1.20 ([43]):

Let (X ; $*$,0) be a BCK-algebra. Then the following are equivalent

(a) $x \leq z$ *and* $z * y \leq z * x$ *imply* $x \leq y$,

(b) $x, y \leq z$ *and* $z * y \leq z * x$ *imply* $x \leq y$,

(c) $x \leq y$ *implies* $x = y * (y * x)$,

(d) X is commutative ,

(e) $x * y = 0$ *implies* $x * (y * (y * x)) = 0$, for all x , y in X .

Definition 1.1.21 ([15]):

If there is an element 1 of a BCK-algebra X satisfying $x \leq 1$ for all $x \in X$, then the element 1 is called a unit of X . A BCK-algebra with unit is said to be bounded. In a bounded BCK-algebra, we denote $1 * x$ by Nx .

Example 1.1.22 ([43]): Let X = { 0, a, b, 1 } and $*$ be given by the table :

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	b	a	0

Then (X ; $*$, 0) is a bounded BCK-algebra and 1 is the unit of X .

Example 1.1.23 ([43]):

Let X = { 0, 1, 2, ….. } with the natural order. Then we define

$$x * y = \begin{cases} 0 \ \ if \ \ x \leq y, \\ 1 \ \ if \ \ y < x \ and \ \ y \neq 0, \\ x \ if \ \ y < x \ and \ \ y = 0. \end{cases}$$

Under the definition of $*$, X is a BCK-algebra, which is not bounded.

Theorem 1.1.24 ([15]):

In a bounded BCK-algebra, we have

(a) $N1 = 0$, $N0 = 1$,

(b) $NNx \leq x$,

(c) $Nx * Ny \leq y * x$,

(d) $y \leq x \Rightarrow Nx \leq Ny$,

(e) $Nx * y = Ny * x$,

(f) $1 \wedge x = x$, $x \wedge 1 = NNx$,

(h) $NNNx = Nx$.

Definition 1.1.25 ([15]):

For a bounded BCK-algebra X , if an element x satisfies $NNx = x$, then x is called an involution.