Ain Shams University
Faculty of Science
Mathematics Department

ON SOME TYPES OF COMPACT SPACES AND NEW CONCEPTS IN TOPOLOGICAL GRAPH THEORY

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Contents

Introduction

In this thesis, we proceed to special properties of topological spaces. We know that compactness and connectedness are two of these. Compactness and connectedness are playing an important role in all branches of mathematics. On the other hand, we introduced new study in topological graph theory so we constructed a topology on the set of vertices of a graph G(V, E) and we studied some properties of this topological space via properties of the graph.

In 1906, the term of "compact" was used for the first time by Frechet. From that time, many sorts of compactness were introduced by different topologists.

In 1968, Asha Mathur [50] described compactness and its weaker forms through a table containing 72 properties.

In 1985, M.E. Abd El-Monsef and A.M. Kozae [2] introduced a property $P_{\alpha\beta\gamma}$ for generalizing 1920 types of compactness and closeness.

This thesis is in continuation to the study of the property $P_{\alpha\beta\gamma}$ of three variables which generalize the notions of compactness, paracompactness, closeness and many of their corresponding weaker forms, via the property $P_{\alpha\beta\gamma}$ we generalized 15456 types of compactness and closeness. We further study some properties of these types and the relationship between various types of compactness was summarized, also.

The other important property of topological spaces which studied in this thesis is connectedness. In this thesis, we introduced some types of connectedness in ideal topological spaces.

The subject of ideals on a nonempty set X has been studied by Kuratowski in 1933 [38]. An ideal *I* on a set X is a nonempty collection of subsets of X which satisfies; (1) A$\in$ *I* and B$\subset$ A implies

B∈ I, (2) A∈ I and B∈ I implies A∪B ∈ I. In 1945, Vaidyanathaswamy [72] introduced the concept of ideal on topological spaces. Given a topological space (X, τ) with an ideal I on X and if P(X) is the set of all subsets of X, a set operator ( )*: P(X)→ P(X), called a local function of A with respect to τ and I, is defined as follows: for A⊂ X, A*(I,τ)= {x∈ X: A∩ U∉ I for every U∈ τ(x)} where τ(x)= {U∈ τ: x∈ U}. A Kuratowski closure operator cl*( ) for a topology τ*(I, τ), called the ⋆-topology, finer than τ is defined by cl*(A) = A∪ A*(I, τ).

In 1992, the term of "α-operator set" in ideal topological spaces was used for the first time by A.A. Nasef [53]. A set operator ( ) $^{\alpha *}$: P(X) → P(X) is called an α-local function of I with respect to τ is defined as follows: for A⊂ X, $A^{\alpha *}(I, \tau)$= {x∈ X: A∩ U∉ I for every U∈ $\tau_\alpha(x)$} where $\tau_\alpha(x)$= {U∈ τ_α: x∈ U}. A α*-closure operator, denoted by $cl^{\alpha *}$( ), for a topology $\tau^{\alpha *}(I, \tau)$ which is called *-α-topology on X finer than τ* (I, τ) is defined as follows: $cl^{\alpha *}(A)$ (I, τ) = A∪ $A^{\alpha *}(I, \tau)$ for every A⊂ X.

In 2002, E. Hatir and T. Noiri [31] introduced "αI-open set" in ideal topological spaces, a subset A of an ideal space (X, τ, I) is said to be αI-open if A⊂ int(cl*(int(A))). The family of all αI-open sets in a space (X, τ, I), denoted by αIO(X), is finer than τ and coarser than $\tau^{\alpha *}(I, \tau)$.

In 2012, the terms of "⋆-connectedness" and "⋆-hyperconnectedness" in ideal topological spaces were introduced for the first time by E. Ekici and T. Noiri [26]. An ideal space (X, τ, I) is called ⋆-connected (respectively ⋆-hyperconnected) if X cannot be written as a disjoint union of a nonempty open set and a nonempty *-open set (respectively if A is ⋆-dense (i.e. cl*(A) = X) for every nonempty open subset A of X).

In this thesis, we introduce the concept of "*I*α-open set" in ideal topological spaces. A subset A of a space X is said to be *I*α-open set if $A \subset int(cl^{\alpha^*}(int(A)))$. The family of all *I*α-open sets in an ideal space (X, τ, *I*), denoted by *I*αO(X), is finer than τ and coarser than α*I*O(X).

So, we precede new characterizations of both ⋆-connectedness and ⋆-hyperconnectedness. Also, we introduce new types of connectedness in ideal topological spaces. Some properties, propositions, remarks and examples to explain these types had been given and the relationships between these types are explained.

In the other part of this thesis, we create new applications on some graphs by using some topological concepts, like; we constructed a topology on the set of vertices of a graph G(V,E) and discussed the relation between two concepts of connectedness in both topological spaces and graphs. So, we used the topology which constructed on the set of vertices of a graph G to determine the definable sub graphs of G, with many corollaries.

This thesis consists of five chapters.

Chapter one consists of four sections. In section one, we recall some basic definitions of some types of near open sets and we explain the relationships among these sets. So we proved that a subset A of a space (X, τ) is β-open [1] if and only if it is weakly open [28]. In section two, we recall some basic definitions, remarks, properties and propositions about compactness on topological spaces. So, we recall the concepts of "net" and "filter" in a topological space, which will be needed later on. In section three, we recall some basic definitions, remarks, properties and propositions about connectedness topological space and in ideal topological space. So, we proceed to other characterization of ⋆-hyperconnectedness. In section four, we recall some basic definitions and concepts in graph theory, which will be needed later on.

Chapter two consists of three sections. In section one, we generalize the concept g-closed set by using some types of near open sets and study the relations between these generalizations. In section two, we shall use the property $P_{\alpha\beta\gamma}$ for generalizing 15456 types of compactness and closeness. So, we discuss the relations between some of these types. In section three, we introduce the notion bg-compactness in the topological spaces. So we study some properties of this type of compactness. We gave many counter examples to show that bg- compact is stronger than both b-compact and g-compact which are stronger than compact. We added some conditions to get the opposite directions.

Chapter three consists of three sections. In section one, we recall and discuss some properties of the concept α-local function in ideal topological spaces which was introduced by A.A. Nasef in 1992 [53]. So, we recall the concept α*I*-open set which was introduced by E. Hatir and T. Noiri in 2002 [31]. Also, we introduce the concept *I*α-open set in an ideal topological spaces and we proved that the family of all *I*α-open sets in a space (X, τ, *I*) is both finer than τ and coarser than the family of all α*I*-open sets in (X, τ, *I*) which is coarser than the family of all α-open sets in a space (X, τ), (briefly, $\tau \subset I\alpha O(X) \subset \alpha I O(X) \subset \tau_\alpha$). In section two, we introduce and study new types of connectedness in ideal topological spaces, like α⋆-hyperconnectedness, α⋆-connectedness, α*I*⋆-connectedness and *I*α⋆-connectedness. So, we give characterizations of these concepts. Also, we study the relations between these types of connectedness. In section three, we introduce the concepts of α⋆-separated sets and $\alpha\star_s$-connectedness in an ideal space (X, τ, *I*). Some properties, propositions, remarks and examples about these concepts are studied.

The relation between $\alpha\star$-separated sets and $\star$-separated sets (respectively between $\alpha\star_s$-connectedness and $\star_s$-connectedness) was explained, also.

Chapter four consists of four sections. In sections one and two; by using two methods we construct a topology on the set of vertices V(G) of an undirected graph G(V,E). In section one, we define the adjacent set of a vertex $v\in$ V(G) to construct a topology on V(G). In section two, we define a neighborhood system on the set of vertices V(G) and using it to construct a topology on V(G). Finally; we discuss the relation between two concepts of connectedness in undirected graphs and in topological spaces, we review some properties, remarks, propositions and examples to explain our study. In sections three and four, we discuss the relation between two concepts of connectedness in directed graphs and in topological spaces. We review some properties, remarks, propositions and examples to explain our study.

Chapter five consists of four sections. In section one, we introduce a new definition of a neighborhood of vertices in a directed graph and used it to define a new concepts of lower and upper approximation of a subgraph H of a directed graph G(V,E). Some properties of these concepts are studied, also. In section two, we define and explain some types of accuracy of approximation of a subgraph H of a graph G(V,E), like accuracy measure, semi-accuracy and pre-accuracy of approximation of a subgraph H. Some properties of these types are studied and some examples to explain these concepts. The relations among these types are studied also. In section three, we introduce the concepts of G-interior and G-closure of a subgraph H of a graph G(V,E), and studied some properties of these concepts. In section four, we use the concepts of G-interior and G-closure to define new types of measures to describe inexactness of approximation of a subgraph H of a graph G(V,E), like G-accuracy, semi-G-accuracy and pre-G-accuracy of approximation of a subgraph H. Some properties of these types are studied and some examples to explain these concepts. The relations among these types are studied also.

Finally, every remark, lemma, proposition and corollary which is not referred to by any reference in this thesis is our own.

Some of the main results in this thesis were published in [3], [4], [55], [56] and [64]. So, some of these results were presented in *The One Day Conference of Discrete Mathematics and Computer Sciences* (2 June 2011), The Egyptian Mathematical Society, The Society Conference of No. 151 was held in Dar al-Diyafa, Ain Shams University, Egypt. And other results were presented in *The 25th Conference of Topology and its Applications* (16 July 2011), Faculty of Science, Port Said University, Egypt. Also, some of new results were presented in *The 26th Conference of Topology and its Applications* (3-4 July 2012), Faculty of Science, Tanta University, Egypt.

CHAPTER ONE

Preliminary Concepts and Results

Preliminary Concepts and Results