Iterative Methods for Solving Singular and Rectangular Systems

Thesis by

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Abstract

The thesis considers the numerical treatment of singular and rectangular linear algebraic systems by iterative techniques. The full rank rectangular system $A\,X = b, A \in R^{m\times n}$, rank(A) = n, is reformulated to introduce square non-singular system of order $m + n$ through partitioning the coefficient matrix into two parts A_1, A_2 with non-singular part A_1. The 2-block KSOR and the 3-block KSOR methods are introduced corresponding to those of the SOR versions. The parameters of the problem are discussed to insure convergence and the selection of their optimal values are considered. Also, the four block SOR method is discussed and the four block KSOR method is introduced for singular systems. Semi convergence properties for iterative methods are discussed. Application of the basic iterative methods (JOR, SOR and KSOR) are applied on the singular system appears in the discretization of Poisson's equation with Neumann and periodic boundary conditions. Moreover, the concepts of preconditioned matrices are considered. Application of the theoretical results and comparison of the performance is considered with numerical examples.

Keywords: SOR, KSOR, 2-block SOR, 3-block SOR, 4-block SOR, rectangular system, singular system, least squares, preconditioned

Summary

Thesis title: "Iterative Methods for Solving Singular and Rectangular Systems"

In general solving systems of linear equations is one of the most important problems in mathematics. Due to the progress in mathematical modelling of realistic problems and their treatments in addition to the difficulties in solving such models exactly, numerical methods becomes the appropriate choice. Systems of linear equations appears in approximately all real models directly or when using approximate methods in solving such models. Non-standard systems of equations appears as a result of the developments in modelling real life problems. Large rectangular linear systems appear in the statistical treatment of data and the problems associated with data fitting by using least squares concepts. Singular linear systems appear in the numerical solutions of differential equations with periodic characters. Recently, iterative techniques for solving large systems of algebraic equations become the common approach due to the progress in computational systems. Using iterative techniques even with standard systems require some rearrangements of the equations to guarantee the convergence of the technique, the situation become more complicated with rectangular and singular systems. Our main objective in this work is the study of rectangular and singular systems and their possible rearrangements to become suitable for using successive iterative techniques. Efficient use of the successive over relaxation method (SOR) requires a good choice of the relaxation parameter. We discussed the choice of the relaxation parameter and we introduced the KSOR versions of the SOR method for the augmented systems appears in the use of the concept of least squares.

This thesis consists of four chapters as follows:

Chapter One: Rectangular and Singular Linear Systems

The basic concepts of rectangular and singular systems are introduced. The generalized invers and their algorithms are used with application to a numerical example. The concept of singular value decomposition is discussed with an algorithm for its calculation with application to a numerical example. The least squares approach is discussed

with its use in solving linear algebraic systems. The Poisson equation are considered with different boundary conditions (Dirichlet, Neumann and periodic boundary conditions). Poisson's equation are transformed to algebraic system form using the five point difference formula.

Chapter Two: Rectangular Linear Systems and Iterative Techniques

An overdetermined linear system is considered with full rank coefficient matrix, $A\,X = b,\; A \in\; R^{m\times n}$, rank(A) = n. The system is reformulated to be suitable for the use of iterative techniques. The 3-block KSOR and the 2-block KSOR versions of the corresponding SOR are introduced. The least squares solution to overdetermined linear system is obtained by using iterative methods (3-block SOR, 3-block KSOR, 2-block SOR and 2-block KSOR). Comparison of the performance of the iterative methods is considered. Discussion of the selection of the relaxation parameters are discussed.

Chapter Three: Singular Linear Systems and Iterative Techniques

The convergence and the semi convergence of iterative methods are established. The optimal iterative methods (SOR, KSOR, and JOR) for solving singular and nonsingular linear systems arising from the numerical treatment of Poisson equation in two dimensions are considered. The iterative methods (SOR, KSOR, and JOR) are applied in the algebraic singular and nonsingular linear systems. The relations for optimum relaxation parameters are established. A comparison between convergence rates are studied.

Chapter Four: Rank Deficient 4 Block and Preconditions

The 4-block SOR method is introduced for the rank deficient overdetermined linear system, the convergence and optimal convergence factors are introduced. The treatment of overdetermined linear system are considered as well as applying the iterative methods (SOR and KSOR) on the normal equation. Two types of preconditioned matrices are applied to the system of normal equations. The comparison between the convergence rates for the all methods are established.

Chapter 1

Rectangular and Singular Linear Systems

Chapter 1 Rectangular and Singular Linear Systems

1.1 Introduction

The main objective of this chapter is to introduce the basic concepts to deal with rectangular and singular linear systems. Precisely we concentrate on the following points

- The Moore Penrose inverse as a generalization to matrix inverse to cover the special cases of the rectangular and singular linear system.
- Study the solution for the overdetermined linear system in generalized concept (least squares sense).
- The singular value decomposition as a basic matrix factorizations to handle the least squares problem.
- Existence of singular and rectangular systems.

1.2 The Moore Penrose Generalized Inverse

It is well known that if the matrix A is nonsingular, then the solution of the linear system $Ax = b$ is given by $x = A^{-1}b$. Also, if the matrix A is singular or rectangular, then the solution of the linear system $Ax = b$ is given by $x = A^{+}b$, where A^{+} is known as the generalized inverse. A generalized inverse or a pseudoinverse A^{+} of a matrix A is a generalization of the inverse matrix to include singular and rectangular matrices. The most widely known type of matrix pseudoinverse is the Moore–Penrose pseudoinverse, which was described by E. H. Moore [18] in 1920. A common use of the pseudoinverse is to compute a least squares solution to an overdetermined linear system. The pseudoinverse is defined for all matrices whose entries are real or complex numbers. The pseudoinverse is unique and can be computed using the singular value decomposition.

In order to define the concept of the generalized matrix inverse for any matrix $A \in C^{m \times n}$ it is important to introduce the following five matrix equations where

1. $AXA = A$
2. $XAX = X$
3. $AX = (AX)^H$, (H denote to the conjugate transpose)
4. $XA = (XA)^H$
5. $XA = AX$

It is clear that if X is the classical inverse the five matrix equations are trivially satisfied. Moreover, only the fifth matrix equation is restricted to square matrices. Now if $\mu \subseteq \{1,2,3,4,5\}$ the matrix $X \in C^{n\times m}$ is defined to be μ-inverse of the matrix A if X satisfy equation (i) where $i \in \mu$. In particular the Moore Penrose generalized inverse of a matrix A is defined to be the {1,2,3,4}- inverse and also called (pseudoinverse) and is denoted by, A^+.

1.2.1 *Important Properties*

1. The Moore Penrose inverse A^+ is unique for any matrix.

   Proof: suppose X, Y are two Moore Penrose inverses of A then

$$X = X(AX)^H = XX^H A^H = X(AX)^H (AY)^H = XAY$$

$$= (XA)^H (YA)^H Y = A^H Y^H Y = (YA)^H Y = Y.$$

2. For any non-singular matrix A , $A^+ = A^{-1}$
3. For any matrix, $(A^+)^+ = A$.
4. $(kA)^+ = (\frac{1}{k})A^+$ and $k \neq 0$.
5. $(A^H)^+ = (A^+)^H$.

   Proof: set $G = A^H$ we will prove that G^+ satisfies the four conditions

$$G^+GG^+ = (A^+)^H A^H (A^+)^H = (A^+AA^+)^H = (A^+)^H = G^+$$

$$GG^+G = A^H (A^+)^H A^H = (AA^+A)^H = A^H = G$$

$$(GG^+)^H = (A^H(A^+)^H)^H = A^+A = (A^+A)^H = A^H(A^+)^H = GG^+$$

$$(G^+G)^H = ((A^+)^H A^H)^H = AA^+ = (AA^+)^H = (A^+)^H A^H = G^+G$$

Then G^+ satisfies all conditions and since Penrose inverse is unique then the proof is complete.

6. $0^+ = 0$.
7. $rank\ (A) = rank(A^+)$.
8. If $P \in C^{m\times m}$ and $Q \in C^{n\times n}$ are unitary matrices and $A \in C^{m\times n}$ then

$$(PAQ)^+ = Q^H A^+ P^H$$

Proof:

let $G = PAQ$ and it is required to prove $G^+ = Q^H A^+ P^H$

$$GG^+G = (PAQ)(Q^H A^+ P^H)(PAQ)$$

$$= PA(QQ^H)A^+(P^H P)AQ = PAIA^+IAQ$$

$$= P(AA^+A)Q = PAQ = G.$$

$$G^+GG^+ = (Q^H A^+ P^H)(PAQ)(Q^H A^+ P^H).$$

$$= Q^H A^+(P^H P)A(QQ^H)A^+P^H = Q^H A^+ IAIA^+ P^H$$

$$= Q^H(A^+AA^+)P^H = Q^H A^+ P^H = G^+.$$

$$GG^+ = (PAQ)(Q^H A^+ P^H) = PA(QQ^H)A^+P^H = PAIA^+P^H$$

$$= P(AA^+)P^H = (GG^+)^H$$

$$G^+G = (Q^H A^+ P^H)(PAQ) = Q^H A^+(P^H P)AQ,$$

$$= Q^H A^+ IAQ = Q^H(AA^+)Q = (G^+G)^H.$$

9. If $A \in C^{n\times k}$ and $B \in C^{k\times m}$ and the two matrices of rank k then

$$(AB)^+ = B^+A^+$$

1.2.2 *Full Rank Factorization*

Consider a nonzero matrix A which has no full rank (column or row), such matrices are known as rank deficient matrices we can factor A into products of two matrices, one has full column rank and the other has full row rank. This factorization is called full rank factorization. Full rank factorization is useful in computing Moore Penrose inverse. The following lemma expresses the factorization.

Lemma 1.1, [1]**:** let $A \in C^{m\times n}$ and $rank\ A = r < \min\{m, n\}$, then there exist matrices $F \in C^{m\times r}$ and $G \in C^{r\times n}$ with $rank\ F = rank\ G = r$ such that

$$A = FG \tag{1.1}$$

Proof: first determine the matrix F such that the columns of F are a basis for the space spanned by the columns of A. So F will be of order $m \times r$ and $rank\ F = r$. Then we can determine the matrix G of order $r \times n$ by the following equation

$$A = FG\ .$$

The matrix G will be unique since each column of A can be represented as a linear combination of the columns of Fin a unique way. Finally, $rank\ G = r$ since

$$rank\ G = r \geq rank\ FG = r$$

In particular, we can choose the columns of F as maximal linearly independent set of columns of A. Also, we can choose the matrix G first such that the rows of G form a basis for the space generated by the rows of A after this determine matrix Fin a unique way by $A = FG$.

In case A has a full rank (column or row) then one of factors will be a unit matrix.

1.2.3 *Compute Moore Penrose Inverse by Full Rank Factorization*

Theorem 1.1, [1]**:** if $A \in C^{m\times n}$and $rank\ A = r\ , r > 0$ and suppose the full rank factorization in (1.1), then

$$A^{+} = G^{H}(F^{H}AG^{H})^{-1}F^{H} \quad (1.2)$$

Proof: first we prove that the matrix $F^{H}AG^{H}$ is non-singular, since $A = FG$ then we get

$$F^{H}AG^{H} = (F^{H}F)(GG^{H})$$

$$(F^{H}AG^{H})^{-1} = (G^{H}G)^{-1}(F^{H}F)^{-1}$$

Clearly both factors have order $r \times r$ and from last lemma both factors have $rank\ r$ and since the product of non-singular matrices is non-singular then $F^{H}AG^{H}$ is non singular

Also

$$G^{H}(F^{H}AG^{H})^{-1}F^{H} = G^{H}(GG^{H})^{-1}(F^{H}F)^{-1}F^{H}$$

The expression $G^{H}(GG^{H})^{-1}(F^{H}F)^{-1}F^{H}$ satisfies four of the stated five matrix equations defining the generalized inverse.

1.2.4 *General Algorithm for Moore Penrose Generalized Inverse*

The generalized inverse for any matrix $A \in C^{m\times n}$ can be calculated according to the following steps

First: Determine the rank of the matrix A and let $rank\ (A) = k$.

Second: Determine a $k \times k$ non-singular submatrix of A.

Third: Use elementary row operations (P) and elementary column operations (Q) of the first kind to move the non-singular sub matrix into the upper left place of A

Accordingly, we find

$$PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad (1.3)$$

where matrix P or Q can be the unit matrix, as a result the matrices A_{12} , A_{21} and A_{22} may not be exist.

Fourth: set $B = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$, $F = A_{11}^{-1} A_{12}$ and $C = [I_k \quad F]$, where I_k the $k \times k$ unit matrix

Fifth: compute the Moore Penrose inverse as follows

$$A^{+} = QC^{H}(CC^{H})^{-1}(B^{H}B)^{-1}B^{H}P. \tag{1.4}$$

A special case when matrix A has linearly independent columns the previous formula is given by

$$A^{+} = (A^{H}A)^{-1}A^{H}. \tag{1.5}$$

As an example: let the matrix

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix},$$

The Moore Penrose inverse for the matrix A can be found as follows,

Apply the previous steps, first we have $rank\ A = 2$, second we get the sub matrix A_{11} where $rank\ A_{11} = 2$ by elimination the second row and the second column,

$$P = Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad , \qquad PAQ = \begin{bmatrix} 2 & -2 & 2 \\ -2 & 6 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

Where

$$A_{11} = \begin{bmatrix} 2 & -2 \\ -2 & 6 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \qquad A_{21} = [2 \quad -2]$$

Then

$$B = \begin{bmatrix} 2 & -2 \\ -2 & 6 \\ 2 & -2 \end{bmatrix} \qquad F = A_{11}^{-1}A_{12} = \begin{bmatrix} 6/8 & 2/8 \\ 2/8 & 2/8 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$