THE COMPLEX REPRESENTATION OF CERTAIN GROUPS

Thesis Submitted in Partial Fulfilment of the Requirements

for

the Award of the M. Sc. Degree

By

KHAIRY KHEIR METAWEIA

Submitted at

Ain Shams University

Faculty of Science

Department of Pure Mathematics

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بسم الله الرحمن الرحيم

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M.SC. COURSES

STUDIED BY THE AUTHOR (1975-1976)

(i)	Theory of Functions of Matrices (during the whole academic year)	2 hours per week
(ii)	The Algebraic Eigenvalue Problem	2 hours per week
(iii)	Functional Analysis	2 hours per week
(iv)	Abstract Algebra	2 hours per week
(v)	Algebraic Topology	2 hours per week

PREFACE

The thesis consists of four chapters.

Chapter I summarizes some facts which will be needed in chapter II.

Chapter II consists of two sections. In the first section we give a review of the complex classical groups while in the second, we investigate three new types of complex representations namely, the complex restricted representation, the complex sphere representation and the complex projective representation. These types of representations can be regarded as extensions of the unitary representations. We have intended that our constructions should be mainly related to the unitary representations because the unitary representations are the most important and most widely studied class of representations. There are two reasons for this. Firstly, unitary representations appear naturally in diverse applications in physics science (dynamical systems, quantum mechanics and the physical theory of fields). Secondly, unitary representations enjoy a series of remarkable properties, which greatly facilitate their study.

Chapter III consists of two sections. In the first section we expose the construction of the $Spin^{c}(n)$-group while in the second, we expose two theorems on the unitary ring of this group.

In chapter IV, we quote some theorems on the representations of a complex Lie group and its Lie algebra. In the first section we consider the representation functions while in the second, we consider the semi-simple representations.

CONTENTS

CHAPTER I

PRELIMINARIES

In this chapter we give some preliminaries which will be needed throughout the thesis.

§1. HOMEOMORPHISM [14]

Definition (1.1.1)

Two topological spaces X and Y are called homeomorphic or topologically equivalent if there exists a bijective (i.e. one-one, onto) function $f : X \longrightarrow Y$ such that f and f^{-1} are continuous. The function f is called a homeomorphism.

Definition (1.1.2)

A topological space X is said to be embedded in a topological space Y if X is homeomorphic to a subspace of Y.

§2. AXIOMS OF COUNTABILITY [14]

Definition (1.2.1)

A topological space X is called a first countable space if it satisfies the following axiom, called the first axiom of countability

$[C_1]$ For each point $p \in X$ there exists a countable class $\mathcal{B}_p$ of open sets containing p such that every open set G containing p also contains a member of $\mathcal{B}_p$.

In other words, a topological space X is a first countable space if and only if there exists a countable local base at every point $p \in X$.

Observe that $[C_1]$ is a local property of a topological space X, i.e. it depends only upon the properties of arbitrary neighborhoods of the point $p \in X$.

Definition (1.2.2)

A topological space $(X, \mathcal{T})$ is called a second countable space if it satisfies the following axiom, called the second axiom of countability.

$[C_2]$ There exists a countable base $\mathcal{B}$ for the topology $\mathcal{T}$.

Observe that second countability is a global rather than a local property of a topological space.

Now if $\mathcal{B}$ is a countable base for a space X, and if $\mathcal{B}_p$ consists of the members of $\mathcal{B}$ which contain the point $p \in X$, then $\mathcal{B}_p$ is a countable local base at p. In other words,,

Proposition (1.2.1)

A second countable space is also first countable.

§3. CENTRALIZER AND NORMALIZER OF A GROUP [4]

Definition (1.3.1)

If G is a group, we define the centre of G, denoted by $C_r(G)$ to be

$\{ c \mid c \in G \text{ and for all } g \in G,\ g\,c = c\,g \}$

$C_r(G)$ turns out to be a normal subgroup of G.

Definition (1.3.2)

The centralizer C(A) of A (in G) is defined by

$C(A) = \{ c \mid c \in G \text{ and for all } a \in A,\ ca = ac \}$.

C(A) is a subgroup of G. If A is an abelian subgroup, A is normal in C (A).

Definition (1.3.3)

The normalizer N(A) of A in G is defined by

$N(A) = \{ n \mid n \in G \text{ and } An = nA \}$.

N(A) is a subgroup of G and if, A is a subgroup of G, A is normal in N(A). Furthermore, if A is a subgroup of G, A is normal in G if and only if N(A) = G.

§4. TOPOLOGICAL GROUPS [11]

Definition (1.4.1)

A topological space G that is also a group is called a topological group if the mapping

$$g_1 : (x, y) \longrightarrow x\,y$$

of G x G onto G is continuous in both variables together and if the inversion mapping

$$g_2 : x \longrightarrow x^{-1}$$

of G onto G is also continuous.

If the group operation is addition instead of multiplication, xy and x^{-1} should be regarded as $x + y$ and $- x$, respectively. The identity of a multiplicative group will be denoted by e and that of an additive group by 0.

§5. MANIFOLDS [3] and [11]

(a) Let f be a real-valued function defined on an open subset S of R^n. Then f is said to be in the class r or C^r, if all mixed partial derivatives of all orders exist and are continuous on S.

(b) Let X be a topological space. An atlas $\mathcal{A}$ of class C^r on X is a collection of pairs (U_α , ψ_α), $\alpha \in A$, an index set, satisfying the following conditions :

(i) Each U_α is an open subset of X and $\{U_\alpha\}$ covers X.

(ii) Each ψ_α is a homeomorphism of U_α on an open subset ψ_α (U_α) of R^n for some fixed n.

(iii) For $U_\alpha \cap U_\beta \neq \phi$,

$$\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta (U_\alpha \cap U_\beta) \longrightarrow \psi_\alpha (U_\alpha \cap U_\beta)$$

is of class C^r.

(iv) Let (U, ψ) be a pair consisting of an open subset U of X and a homeomorphism ψ of U onto an open subset of R^n. If for each pair (U_α , ψ_α) $\in \mathcal{A}$ for which $U_\alpha \cap U \neq \phi$ the map

$$\psi \circ \psi_\alpha^{-1} : \psi_\alpha (U \cap U_\alpha) \longrightarrow \psi (U \cap U_\alpha)$$

is of C^r, then (U, ψ) $\in \mathcal{A}$

(c) A Hausdorff topological space X with an atlas $\mathcal{A}$ is called an n-dimensional C^r manifold or simply a manifold. The members of the atlas are usually called the coordinate system for the manifold.

If the mapping f in the above definition has a power series expansion, then f is said to be analytic.

Definition (1.3.1)

A Lie group G is a set with the two structures :

(i) a group structure,

(ii) a real (or complex) analytic manifold, and these are compatible with one another, i.e. the equations defining the group operations are analytic in the coordinates of the manifold. We speak of a real or complex Lie group. Of course any complex Lie group is also a real Lie group by the realization method [7].

Definition (1.3.2)

A vector at a point p of a real analytic manifold is a mapping L of A (p) (the analytic functions defined in the neighborhood of p) into R such that

(i) L is R - linear,

i.e. L (a f + b g) = a L (f) + b L (g), a,b $\in$ R.

(ii) L is a differentiation,

i.e. L (fg) = f(p) L (g) + g (p) L (f).

Definition (1.3.3)

A vector field (or infinitesimal transformation) on a manifold is a mapping $p \longrightarrow T(p)$ ($T(p)$ denotes the vector space of tangent vectors at p). A vector field X is analytic in U if the function $X(p)\, f$ is analytic for every analytic function f in U.

A vector field X can be written locally

$$X = \sum_i X_i \frac{\partial}{\partial x_i}, \quad X_i \text{ analytic functions of } (x_i).$$

If X and Y are two (analytic) vector fields, then in general XY is not a vector field. In fact, let $x_1, \ldots, x_n$ be coordinates and let

$$X = \sum_i X_i \frac{\partial}{\partial x_i}, \quad Y = \sum_i Y_i \frac{\partial}{\partial x_i},$$

then

$$XY(f) = \sum_i X_i \frac{\partial}{\partial x_i} \sum_j Y_j \frac{\partial f}{\partial x_j}$$

$$= \sum_{i,j} X_i Y_j \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_j \left\{ \sum_i X_i \frac{\partial Y_j}{\partial x_i} \right\} \frac{\partial f}{\partial x_j}.$$

These equations show at once that $XY - YX$ is a vector field given by

$$XY - YX = \left(\sum_i X_i \frac{\partial Y_j}{\partial x_i} - \sum_i Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}.$$

This operation on vector fields is denoted by a bracket $[X, Y]$. Clearly from its definition it has the following properties :

(i) $[X, Y] = - [Y, X]$

(ii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

(Jacobi identity)

(iii) It is bilinear over R.

Any vector space over R with a bracket satisfying these relations is called a Lie algebra over R (possibly infinite dimensional). Thus the vector fields on a real analytic manifold form a Lie algebra (of infinite-dimension). But we shall only consider finite dimensional Lie algebras.

Now let G be a real Lie group, and let L be the tangent space at the identity. Then, by left multiplication we obtain a left invariant vector field on G corresponding to every vector of L. Using the bracket operation for vector fields we obtain therefore a bracket operation in L. The Lie algebra thus obtained is called the Lie algebra of G, or the infinitesimal group of G, and is basic to the study of G.

<u>Remark</u> : We could have used right invariant vector fields to define the Lie algebra structure, but the map

$$g \longrightarrow g^{-1}$$