ON G-FUNCTIONS AND ON WHITTAKER FUNCTIONS

Thesis Submitted In Partial Fulfilment Of
The Requirements For The Award Of The
Degree Of M. Sc.

By

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1975

PREFACE

The thesis deals with the G-function and Whittaker function. The importance of the G-function derives largely from the possibility of expressing by means of the G-symbol a great many of the special functions, so that each of the formulae developed for the G-function becomes a most key formula from which a very large number of relations can be deduced for Bessel functions, Whittaker functions, E-functions and other related functions.

The results obtained are grouped in three chapters.

In chapter I, we evaluate two main integrals involving G-functions in terms of G-functions. Using these two integrals we sum a number of series involving products of G-functions.

In Chapter II, we give formulae involving summations of G-functions representing Whittaker functions, formulae involving summations of G-functions representing products of Whittaker's functions and formulae involving summations of G-functions representing Bessel functions.

In chapter III, we evaluate integrals involving Whittaker functions and Bessel functions in terms of G-functions. Giving appropriate values to the parameters

in these formulae we deduce some integrals involving Whittaker functions and Bessel functions.

I wish to express my deepest gratitude and thankfulness to Dr. M.A. Simary, Assistant prof. at Cairo High Institute of Technology, Helwan, for his kind supervision and involuable help during the preparation of the thesis.

I should like also to express my sincerest thanks to Prof. Dr. Ragy H.Makar for his kind help and encouragement.

February 1975

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C O N T E N T S

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M.Sc. Courses Studied By The Author

1972 / 1973

Theory of functions of matrices (during the whole academic year)	2 hours weekly
Abstract Algebra	2 hours weekly
Numerical analysis	2 hours weekly
Functional analysis	2 hours weekly
Special functions	2 hours weekly

CHAPTER 0

I N T R O D U C T I O N

In this introductory chapter we give the definition of the hypergeometric function, the generalized hypergeometric function, the Bessel function, the Whittaker function, the E-function and the G-function.

0.1: The hypergeometric function

$$(1) \quad {}_2F_1\,(a,b;c:Z) = \sum_{n=0}^{\infty} \frac{(a;n).(b;n)}{n!\,(c;n)}\, Z^n$$

where

$$|Z| < 1 \qquad \text{and } (a;n) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

satisfies the differential equation

$$Z\,(1-Z)\,\frac{d^2w}{dZ^2} + \left[c-(a+b+1)\ Z\right] \frac{dw}{dZ} - abw = 0.$$

It can be represented as an integral; we have when

$|Z| < 1$ and $\operatorname{Re} c > \operatorname{Re} b > 0$.

$$(2) \quad {}_2F_1\,(a,b;c:Z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-Zt)^{-a}\,dt.$$

When $\operatorname{Re}\,(c-a-b) > 0$ we have from (2).

$$(3) \quad {}_2F_1\,(a,b;c:1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

which is Gauss theorem.

When $|Z| < 1$, $|1-Z| < 1$ and provided $\operatorname{Re}\,(a+b) \leq \operatorname{Re} c < 1$, (1) can be written in the form.

$$(4) \quad {}_2F_1\,(a,b;c:Z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, {}_2F_1(a,b;1+a+b:1-Z)$$

$$+ \frac{\Gamma(c)\Gamma(\text{[illegible]})}{\Gamma(a)\Gamma(\text{[illegible]})}\,\text{[illegible]}\,{}_2F_1\,\text{[illegible]}\,1+\text{[illegible]}-a-b\text{; }1-Z).$$

If Re $c >$ Re $b > o$ then

$$(5) \quad {}_2F_1\,(a,b,b-a+1:-1) = \frac{\Gamma(1+b-a)\,\Gamma(1+\frac{1}{2}b)}{\Gamma(1+b)\,\Gamma(1+\frac{1}{2}b-a)}$$

which is Kummer's theorem.

If $|Z|<1$ and Re $c >$ Re $b > o$, then

$$(6.a) \quad {}_2F_1\,(a,b;c:Z) = (1-Z)^{-a}\;{}_2F_1\left(a,c-b;c:\frac{Z}{Z-1}\right),$$

$$(6.b) \quad {}_2F_1\,(a,b;c:Z) = (1-Z)^{-b}\;{}_2F_1\left(b,c-a;c:\frac{Z}{Z-1}\right).$$

0.2: Barnes contour integral for ${}_2F_1\,(a,b;c:Z)$

Barnes, following Pincherle and Mellin has represented the hypergeometric function by means of a contour integral.

A particular case of Barnes formula is

$$(7) \quad -\frac{1}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i} \frac{\pi}{\sin \pi s}\,(-Z)^{s}\,ds = 1+Z+Z^{2}+\ldots$$

valid when $|Z|<1$ and $|\arg(-Z)|<\pi$, the path of integration being the straight line Re $s = \sigma$ where $-1<\sigma<o$.

Barnes contour integral for ${}_2F_1\,(a,b;c:Z)$ is

$$(8) \quad \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}\cdot{}_2F_1\,(a,b;c:Z)$$

$$= \frac{1}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i}\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}\cdot\Gamma(-s)\,(-Z)^{s}\,ds.$$

${}_1F_1\,(a;c:Z)$ can be defined from ${}_2F_1\,(a,b;c:Z)$ as follows:

$${}_1F_1\,(a;c:Z) = \lim_{b\to o}\;{}_2F_1\left(a,b;c:\frac{Z}{b}\right).$$

The function ${}_1F_1$ (a;b:Z) satisfies the differential equation

$$Z \frac{d^2w}{dZ^2} + (b-Z) \frac{dw}{dZ} - aw = 0.$$

When $|\arg (-Z)| < \frac{\pi}{2}$ and a is not a negative integer or zero, then

$$(9) \quad {}_1F_1 (a;b:Z) = \frac{\Gamma(b)}{\Gamma(a)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma(a+s)\Gamma(-s)(-Z)^s}{\Gamma(b+s)} ds.$$

0.3: The generalized hypergeometric function

The generalization of ${}_2F_1 (a,b;c:Z)$ can be made by the introduction of P parameters of the nature of a , b and q parameters of the nature of c; thus

$$(10) \quad {}_pF_q \begin{pmatrix} a_1, a_2, \ldots, a_p : Z \\ b_1, b_2, \ldots, b_q \end{pmatrix} = {}_pF_q (a_r, b_t : Z)$$

$$= \sum_{n=o}^{\infty} \frac{(a_1;n)(a_2;n)\ldots(a_p;n)}{(b_1;n)(b_2;n)\ldots(b_q;n)\; n!} Z^n.$$

(10) is known as the generalized hypergeometric series, ${}_pF_q$ converges for all finite Z if $p \leq q$, converges for $|Z|<1$ if $p = q + 1$, and diverges for all $Z \neq 0$ if $P > q+1$.

We note that

$${}_pF_q \begin{pmatrix} a_1, a_2, \ldots, a_p : Z \\ b_1, b_2, \ldots, b_q \end{pmatrix}$$

satisfies the differential equation.

$$(11) \quad \theta(\theta + b_1 - 1)(\theta + b_2 - 1)\ldots(\theta + b_q - 1)w$$

$$= Z(\theta + a_1)(\theta + a_2)\ldots(\theta + a_p)w$$

where

$$\theta = Z \frac{d}{dZ} .$$

0.4 : Definition of Bessel function

Bessel function is the solution of the differential equation

$$Z^2 \frac{d^2 w}{dZ^2} + Z \frac{dw}{dZ} + (Z^2 - n^2)\, w = 0;$$

n and Z can be complex.

Special solutions of this equation are the Bessel Neumann and Hankel functions, $J_n(Z)$, $Y_n(Z)$, $H_n^{(1)}(Z)$, $H_n^{(2)}(Z)$ respectively. The latter three are linear combinations of the first. They are defined by

$$(12) \qquad J_n(Z) = \sum_{m=0}^{\infty} \frac{(-)^m}{m!\,\Gamma(n+m+1)} \left(\frac{Z}{2} \right)^{n+2m}$$

$$= \frac{\left(\frac{Z}{2} \right)^n}{\Gamma(n+1)} \; {}_0F_1 \left(n+1; -\frac{Z^2}{4}\right).$$

$$(13) \qquad Y_n(Z) = \left[\sin (n\pi) \right]^{-1} \left[J_n(Z) \cos (n\pi) - J_{-n}(Z) \right].$$

$$(14) \qquad H_n^{(1)}(Z) = J_n(Z) + i\, Y_n(Z) = (i \sin (n\pi))^{-1} \left[J_{-n}(Z) - J_n(Z)\, e^{-i\pi n} \right].$$

$$(15) \qquad H_n^{(2)}(Z) = J_n(Z) - iY_n(Z) = (i \sin(n\pi))^{-1} \left[J_n(Z) . e^{i\pi n} - J_{-n}(Z) \right] .$$

In case n is an integer or zero the right hand sides of the above equations become indeterminate.

Modified Bessel functions

[illegible]

modified Bessel differential equation

$$Z^2 \frac{d^2w}{dZ^2} + Z \frac{dw}{dZ} - (Z^2 + n^2) \; w = 0.$$

The modified Bessel differential equation is obtained by replacing Z by iZ in Bessel's differential equation.

Special solutions are

$$(16) \qquad I_n(Z) = e^{-i\frac{\pi}{2}n} J_n(Z e^{i\frac{\pi}{2}}) = \sum_{m=0}^{\infty} \frac{(\frac{Z}{2})^{n+2m}}{m!\,\Gamma(n+m+1)}$$

$$= \frac{(\frac{Z}{2})^n}{\Gamma(n+1)} \; {}_0F_1 \left(n + 1 : \frac{Z^2}{4}\right).$$

$$(17) \qquad K_n(Z) = \frac{1}{2}\pi \left[\sin (n\pi) \right]^{-1} \left[I_{-n}(Z) - I_n(Z) \right]$$

or

$$K_n(Z) = \frac{1}{2}\pi \left[\sin (\pi n) \right]^{-1} \left[e^{i\frac{\pi}{2}n} J_{-n}(Ze^{i\frac{\pi}{2}}) - e^{-i\frac{\pi}{2}n} J_n(Ze^{i\frac{\pi}{2}}) \right]$$

or expressed in terms of Hankel's functions

$$(18) \qquad K_n(Z) = \frac{1}{2}\, i\pi\, e^{i\frac{\pi}{2}n} H_n^{(1)}(Ze^{i\frac{\pi}{2}}) = -\frac{1}{2}\, i\pi\, e^{-i\frac{\pi}{2}n} H_n^{(2)}(Ze^{-i\frac{\pi}{2}}).$$

We have:

$$K_{-n}(Z) = K_n(Z)$$

$$Y_n(Ze^{i\frac{\pi}{2}}) = i\, e^{in\frac{\pi}{2}} I_n(Z) - \frac{2}{\pi} e^{-in\frac{\pi}{2}} K_n(Z)$$

$$(19) \qquad K_n(Ze^{i\frac{\pi}{2}}) = \frac{1}{2}\, i\pi\, e^{i\frac{\pi}{2}n} H_n^{(1)}(Ze^{i\pi}) = -\frac{1}{2} i\pi\, e^{-in\frac{\pi}{2}} H_n^{(2)}(Z).$$

0.5: Definition of Whittaker function

We have the Kummer's differential equation

$$(20) \qquad Z \frac{d^2y}{dZ^2} + (\text{[illegible]} - Z) \frac{dy}{dZ} \; \text{[illegible]}$$

If we put $y = e^{Z/2} \cdot Z^{-c/2} u$, we get

$$(21) \quad \frac{d^2u}{dZ^2} + \left[-\frac{1}{4} + \frac{\frac{c}{2} - a}{Z} + \frac{\frac{c}{2}\left(1-\frac{c}{2}\right)}{Z^2} \right] u = 0.$$

Again taking $\frac{c}{2} - a = k$ and $\frac{c}{2}\left(1 - \frac{c}{2}\right) = \frac{1}{4} - m^2$

equation (21) takes the form

$$(22) \quad \frac{d^2u}{dZ^2} + \left(-\frac{1}{4} + \frac{k}{Z} + \frac{\frac{1}{4} - m^2}{Z^2} \right) u = 0.$$

The solutions of equation (22) in the neighborhood of the point at infinity are

$$(23) \quad M_{k,m}(Z) = e^{-\frac{Z}{2}} Z^{+m+\frac{1}{2}} {}_1F_1 \left(m + \frac{1}{2} - k;\ 1 + 2m : Z\right)$$

$$(24) \quad M_{k,-m}(Z) = e^{-\frac{Z}{2}} Z^{-m+\frac{1}{2}} {}_1F_1 \left(-m+\frac{1}{2}-k;\ 1-2m : Z\right).$$

(23) and (24) form a system of linearly independent solutions of equation (22) provided that $m \neq -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$.

$$(25) \quad W_{k,m}(Z) = \frac{\Gamma(2m)}{\Gamma\left(m+\frac{1}{2}-k\right)} M_{k,-m}(Z) + \frac{\Gamma(-2m)}{\Gamma\left(-m+\frac{1}{2}-k\right)} \cdot M_{k,m}(Z).$$

$W_{k,m}(Z)$ is called Whittaker function.

The values of $W_{k,m}(Z)$ when Z lies outside the principal branch, can be expressed in terms of the values in the principal branch by means of:

$$(26) \quad W_{k,m}\left(Ze^{\pm i\pi}\right) = \frac{\Gamma(2m)}{\Gamma\left(m+\frac{1}{2}-k\right)} e^{\pm i\frac{\pi}{2}(1-2m)} M_{-k,-m}(Z)$$

$$+\Gamma(-2m)\, e^{\pm i\frac{\pi}{2}(1+2m)}\, M_{-k,m}(Z)$$

$$=\frac{\Gamma(m+\frac{1}{2}+k)}{\Gamma(m+\frac{1}{2}-k)}\, e^{\pm i\frac{\pi}{2}(1-2m)}\, W_{-k,m}(Z)$$

$$+\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)}\, e^{\pm i\frac{\pi}{2}(1+2m)}\left[1-\frac{\cos\pi(m-k)}{\cos\pi(m+k)}\, e^{\pm i2\pi m}\right]M_{-k,m}(Z).$$

<u>Some recurrence relations for $W_{k,m}(Z)$</u>

(27) $$(m-\tfrac{1}{2}+k)Z^{\frac{1}{2}}\, W_{k-\frac{1}{2},m-\frac{1}{2}}(Z)-(2m+Z)\, W_{k,m}(Z)+Z^{\frac{1}{2}}\, W_{k+\frac{1}{2},m+\frac{1}{2}}(Z)=0$$

(28) $$Z^{\frac{1}{2}}\, W_{k+\frac{1}{2},m-\frac{1}{2}}(Z)-2m\, W_{m,k}(Z)+Z^{\frac{1}{2}}\, W_{k+\frac{1}{2},m+\frac{1}{2}}(Z)=0$$

(29) $$(m+k)W_{k-\frac{1}{2},m}(Z)-Z^{\frac{1}{2}}\, W_{k,m+\frac{1}{2}}(Z)+W_{k+\frac{1}{2},m}(Z)=0$$

(30) $$(\tfrac{1}{2}+m-k)(\tfrac{1}{2}-m-k)W_{k-1,m}(Z)+(2k-Z)W_{k,m}(Z)+W_{k+1,m}(Z)=0$$

<u>Some special cases of $W_{k,m}(Z)$</u>

(31) $$W_{0,\frac{1}{2}}(Z)=e^{-Z/2}$$

(32) $$W_{0,\frac{1}{2}}(-Z)=e^{Z/2}$$

(33) $$W_{o,\frac{1}{2}}(\pm i Z) = e^{\pm iZ/2}$$

(34) $$W_{m+\frac{1}{2},m}(Z) = W_{m+\frac{1}{2},-m}(Z) = \exp\left(-\frac{Z}{2}\right) Z^{m+\frac{1}{2}}$$

Integral representation of $W_{k,m}(Z)$

(35) $$\Gamma\left(\tfrac{1}{2}+m-k\right). W_{k,m}(Z) = \exp\left(-\frac{Z}{2}\right) Z^{k+\frac{1}{2}}$$

$$. \int_{o}^{\infty} e^{-Zt} \; t^{m-k-\frac{1}{2}} \; (1+t)^{m+k-\frac{1}{2}} \, dt$$

where $|\arg Z| < \pi/2$ and $\operatorname{Re}(m-k) > -\frac{1}{2}$.

(36) $$\Gamma\left(\tfrac{1}{2}+m-k\right) W_{k,m}(Z) = \exp\left(\frac{Z}{2}\right). Z^{m+\frac{1}{2}} \int_{1}^{\infty} e^{-Zt} \; t^{m+k-\frac{1}{2}} (t-1)^{m-k-\frac{1}{2}} dt$$

where $|\arg Z| < \pi/2$ and $\operatorname{Re}(m-k) > -\frac{1}{2}$.

In terms of a Barnes type integral, the function $W_{k,m}(Z)$ is given by

(37) $$\Gamma\left(\tfrac{1}{2}+m-k\right) \Gamma\left(\tfrac{1}{2}-m-k\right). W_{k,m}(Z) = \frac{1}{2\pi i} \exp\left(-\frac{Z}{2}\right) Z^{k}$$

$$. \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(-t)\, \Gamma\left(\tfrac{1}{2}+m-k+t\right) \Gamma\left(\tfrac{1}{2}-m-k+t\right) (Z)^{-t} \, dt$$

where

$$|\arg Z| < \frac{3\pi}{2} \text{ and } \frac{1}{2}+k \pm m \neq 0, 1, 2, \ldots$$

the path of integration is to be chosen in such a way as to separate the poles of

$$\Gamma(-t), \Gamma(\tfrac{1}{2} + m - k + t) \text{ and } \Gamma(\tfrac{1}{2} - m - k + t).$$

0.6: Definition of the E-function:

Macrobert's E-function arose from an attempt to give a meaning to the symbol, ${}_pF_q$ when $p > q + 1$. For $p \leqslant q + 1$ we have

$$(38) \quad E(p, a_r; q, b_s : X) = \frac{\Gamma(a_1)\ldots\Gamma(a_p)}{\Gamma(b_1)\ldots\Gamma(b_q)} \cdot {}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q : -\frac{1}{X}\right)$$

where $p \ngtr q$ and $X \neq 0$; formula (38) also holds when $p = q + 1$ provided that $|X| > 1$.

$$(39) \quad E(p, a_r; q, b_s : X) = \sum_{r=1}^{p} \frac{\prod_{s=1}^{p}{}' \Gamma(a_s - a_r)}{\prod_{t=1}^{q} \Gamma(b_t - a_r)} \Gamma(a_r) X^{a_r} \cdot {}_{q+1}F_{p-1}\left(\begin{matrix} a_r, a_r - b_1 + 1, \ldots, a_r - b_q + 1 : (-)^{p+q}(X) \\ a_r - a_1 + 1, \ldots, *, \ldots, a_r - a_{p+1} \end{matrix}\right)$$

where $p > q+1$; formula (39) also holds when $p = q + 1$ provided that $|X| < 1$. The prime in $\prod'$ indicates the omission of the factor $\Gamma(a_r - a_r)$, the asterisk in F indicates the omission of the parameter $a_r - a_r + 1$, an