Ain Shams University
Faculty of Science
Mathematics Department

ON GRACEFUL GRAPHS AND SOME SORT OF COLOURING

Thesis
Sumbitted in partial fulfillment
of the requirement for
The M.Sc. Degree in Mathematics

BY

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On Graceful Graphs and Some Sort of Colouring

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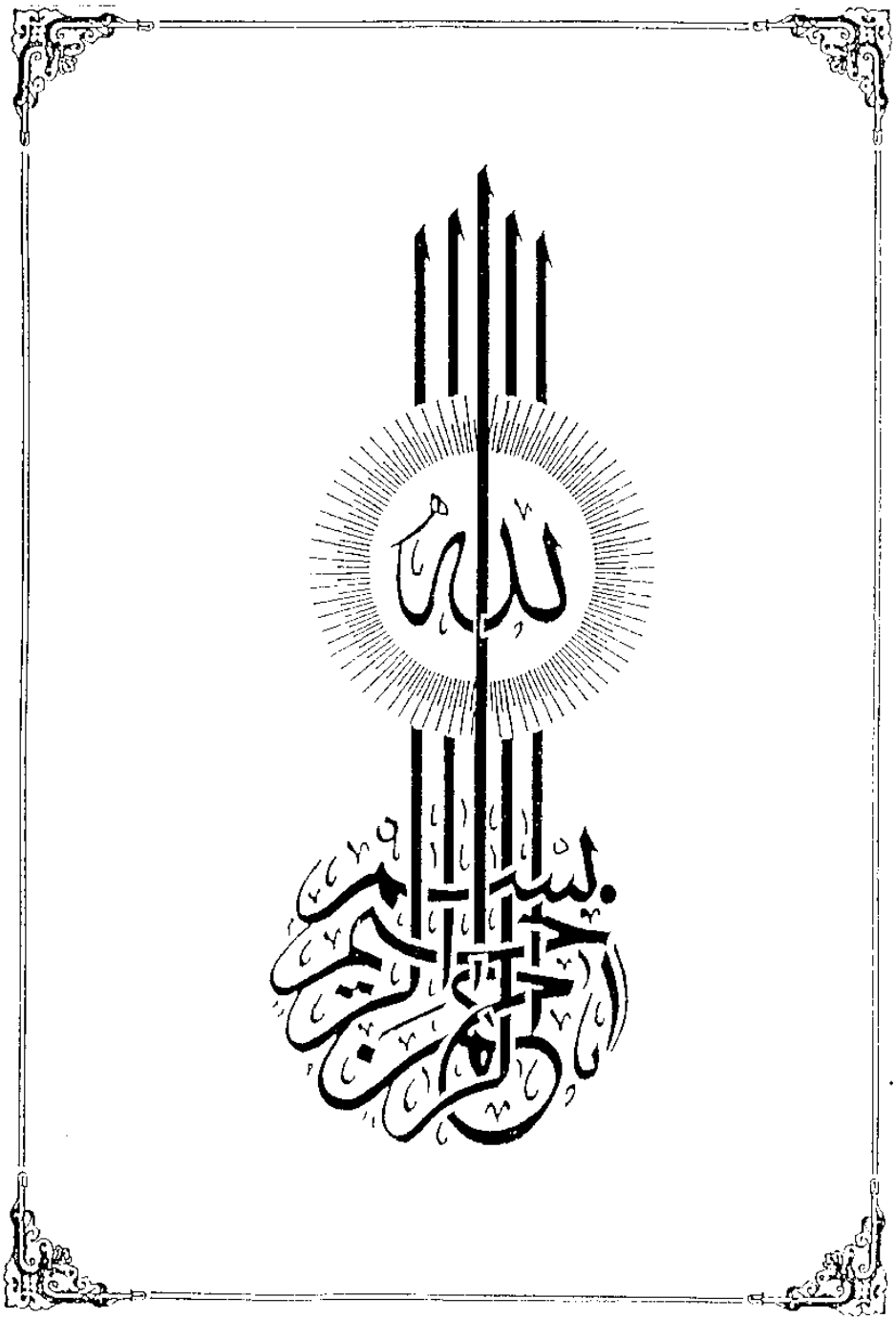

بسم الله الرحمن الرحيم

(... قال رب اشرح لى
صدرى ويسر لى امرى
واحلل عقدة من لسانى
يفقهوا قولى ...)

صدق الله العظيم

سورة طه آيه ٢٥-٢٧

M.Sc. Courses

Beside the work carried in this thesis, the candidate has attended graduate courses for one year in Pure Mathematics, covering the following topics:

1. Abstract Algebra.
2. Algebraic Topology.
3. Functional Analysis.
4. Mathematical Analysis.
5. Theory of Groups.

He has successfully pased a written examination in these courses (1990-1991).

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CONTENTS

ABSTRACT

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ABSTRACT

In this thesis, we obtain new results on two different topics pertaining the labeling of graphs, on one hand is the harmonious labeling of the cube of the path, P_n^3 and also the disjoint union C_nUC_{n+1}, and on the other hand the total chromatic number of some cartesian products of two graphs and also of the cube of the path, P_n^3.

In order to study the above requirements we must give the definitions of:

(i) graceful and harmonious labeling of graphs, and

(ii) total colouring of graphs.

SUMMARY

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This thesis is concerned with two very important topics in Graph Theory. The first one is the labeling of graphs, namely the graceful and harmonious labeling.

In 1966 Rosa defined the graceful labeling and he called it the β-valuation of graphs. In 1980 Graham and Sloane defined the harmonious labeling. The second topic is the total colouring of graphs, which had been independently introduced by Behzad (1965) and Vizing (1964) .

The thesis consists of four chapters : The first one represents a review of the well-known results in graceful and harmonious labeling.. The second one contains new results and the material of it has been accepted for publication in the Canadian Journal "Utilitas Mathematica" .

The third chapter is again a review but for well-known results in total colouring. The last chapter deals with the total colouring of Cartesian products of some families of graphs and the cube of a path. The material of this chapter is also new, and a great deal of it is submitted for publication.

CHAPTER 1
BASIC CONCEPTS AND RESULTS CONCERNING GRACEFUL AND HARMONIOUS GRAPHS

CHAPTER 1

Basic Concepts and Results Concerning Graceful and Harmonius Graphs

1.1 Introduction

A graph G = (V, E) is numbered if each vertex v is assigned a non - negative integer $\phi(v)$, and each edge vw is assigned the absolute value of the difference of the numbers at its endpoints - that is, $|\phi(v) - \phi(w)|$. This numbering is called graceful, also called β-valuation [16], if, furthermore, we have:

(a) the vertices are labled with distinct integers (that is, ϕ is an injection);

(b) the largest value of the vertex - labels is equal to the number of edges (that is, $\max_{v \in V} \phi(v) = |E| = e$);

(c) all the edges of G have distinct labels chosen from the set $\{1,2,3,\ldots,e\}$.

A graph which admits a graceful numbering is said to be a graceful graph, e.g., let K_n and C_n denote the complete graph on n vertices and the cycle of length n, respectivelv. The graceful numbering of K_3, K_4, K_5-e, and C_4 are described in Figure 1.1 .

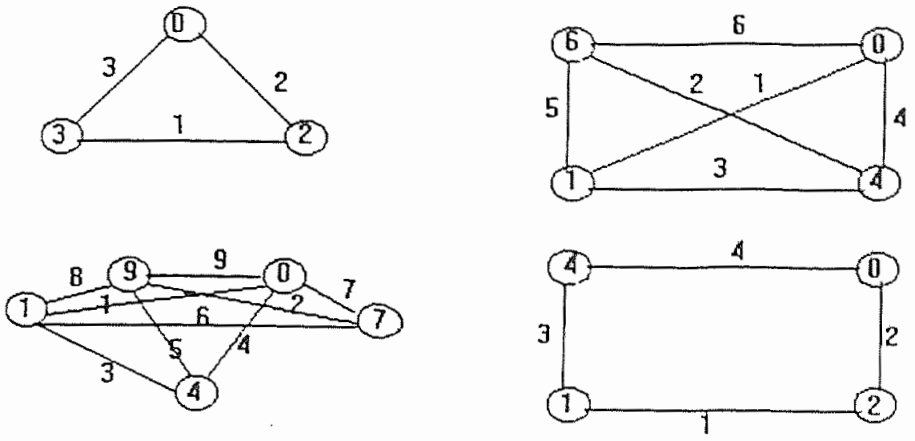

Fig . 1.1

We have to note here that not all graphs are graceful for example ,the disjoint union of two complete graphs K_2 and K_3 is not graceful, we can show this as follows :

Since |E| = e = 4, it follows that the vertex labels 0 and 4 must occur on one component (either K_2 or K_3) to induce the edge label 4 (=e). Since also the vertex labels are all distinct, Figure 1.2 exhibits all the possible labelings of vertices of the graph $K_2 \cup K_3$ and their induced edge labels are not all distinct, then these labelings are not graceful .

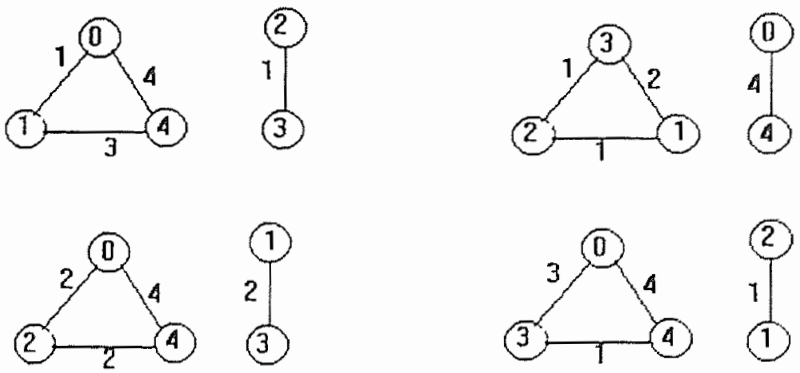

Fig. 1.2

The main unsolved problem is to determine which graphs are graceful . This appears to be very difficult , one reason being that a subgraph of a graceful graph need not be graceful (for example , C_5 is not graceful, although it is a subgraph of the graph $K_5 - e$, which is shown to be graceful in Figure 1.1) .

Our object here is to describe the current state of knowledge about the gracefulness of some classes of graphs, which they are also arisen in many applications (for example , coding theory , X- ray crystallography , radar , communication networks and astronomy).

1.2 Graceful trees .

Kotzig in Rosa's article [16] has made the conjecture that all trees are graceful . Despite many efforts , this conjecture has not been proved ; however , many classes of trees have been shown to be graceful , and a wide literature exists on the subject . We now give the following result .

Theorem 1.2.1

All paths P_n , $n > 2$ are graceful .

Proof. A simple graceful labeling of paths P_n is shown in Figures 1.3 (i) and 1.3 (ii), for the two cases $n = 2k$ and $n=2k+1$, $k \geq 1$, respectively .