ON THE ZEROS OF STATIONARY GAUSSIAN PROCESSES

A THESIS

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by

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M.Sc. COURSES

STUDIED BY THE AUTHOR (1972-73)

1- Theory of Probability	2 hrs. weekly
2- Theory of Storage	2 hrs. weekly
3- Stochastic Processes	2 hrs. weekly
4- Design of Experiments	2 hrs. weekly
5- Statistical Inference	2 hrs. weekly

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SUMMARY

The thesis aims at investigating the probabilities of zeros of random processes and their characteristics which have many applications in communication theory. It also aims at preparing computer program for the probability density function of the interval between m zeros of the normal random process.

The thesis is composed of an introduction and four chapters. The first chapter introduces the probability distribution function of zeros and their expected number for a random process.

In the second chapter we discuss the probability density function of the interval between m zeros for the case of a general random process. Also the probability of exactly m zeros in a small time interval is investigated.

In the third chapter, we considered the normal random process, i.e. the Gaussian process, and obtained the desired probability (the probability of the interval between m zeros) in a form of an infinite series. The first three terms are easily evaluated. The fourth term involves a four-variate integral, which can not be expressed in a

closed form. We reduced this integral to a double integral that is more suitable for numerical computations. Then we wrote a Fortran program to Compute this integral.

Finally in chapter four, we considered a random process with a special covariance function, and obtained the probability density function of the interval between any two successive zeros.

INTRODUCTION

The problem of investigating the probabilities of zeros and related characteristics of random processes was first brought into attention by Rice [11]. Knowledge of these characteristics in communication theory helps to control the noise associated with the transmission of signals.

Longuet-Higgins [6] obtained an infinite series representation for the probability density of the interval between m zeroes of a general random process. Wong [12] obtained such a probability for a random process with a special covariance function. Others also obtained approximate expressions for such probability (see Blake [1]).

The first chapter deals mainly with Rice's model. The expected number of zeros in the interval (t_1, t_2) are obtained for a wave function with random amplitudes in the form

$$\int_{t_1}^{t_2} dt \int_{-\infty}^{\infty} |\eta| \quad P(o, \eta; t) d\eta \quad ,$$

where $P(\xi, \eta; t)$ is the probability density function for the two variables

$$\xi = F(a_1, a_2, \ldots, a_N; t)$$

$$\text{and} \quad \eta = \frac{\partial F}{\partial t} \quad ,$$

where $a_1, a_2, \ldots, a_N$ are random variables.

Also in this chapter the problem of determining the distribution function for the distance between two successive zeros is discussed.

In chapter two, we obtained the probability density of the interval τ between the $i^{\underline{th}}$ and $(i+m+1)^{\underline{th}}$ zeros of the general random process f(t), which was investigated by Longuet-Higgins [6], in the form

$$P_m(\tau) = (m+1) \sum_{i=0}^{\infty} (-)^i \frac{m+2i!}{i!\,m+i+1!} X_{m+2+2i;\,n+1+2i} ,$$

where

$$X_{n,s} = \iint\cdots\int_{t_1 < t_2 < \cdots < t_n} \frac{W(S)}{W(+)}\, dt_2 dt_3 \cdots dt_{n-1} ,$$

which represents the conditional probability that each of the (n-1) intervals (t_i, t_i+dt_i), $i = 2,3,\ldots,n$; contain zero-crossing with the appropriate sign up-crossing or down-crossing (i.e. plus or minus) of the random process f(t), given that it has an up-crossing at t_1. S denote any sequence of n signs, plus or minus, the first being plus, and s denote the number of times that the sequence S changes sign.

Another form of $P_m(\tau)$ due to Rice [11] is also obtained in the form

$$P_m(\tau) = \sum_{i=0}^{\infty} (-)^i \binom{m+i}{i} Y_{m+2+i} ,$$

where Y_n is related to $X_{n,s}$ by the relation

$$Y_n = \sum_{s=o}^{n-1} \binom{n-1}{s} X_{n,s} \quad .$$

Also, in chapter two, the probability of exactly n zeros in a small time interval is investigated.

In chapter three we apply Longuet-Higgin's model to an important special case of a random process, i.e. the normal random process denoted by X(t). The importance of this special case stems from its applicability in communication theory.

The treatment of the normal random process leads to an infinite series, where the $n^{\underline{th}}$ term of that series involves the integral

$$J_n = \int_o^{\infty} \int_o^{\infty} \cdots \int_o^{\infty} x_1 x_2 \ldots x_n \; Z(\underline{x}, \underline{\nu}) \; dx_1 \; dx_2 \ldots dx_n \; ,$$

where $Z(\underline{x}, \underline{\nu})$ is the ordinary normal probability density in the n variables x_i, with the covariance matrix $(\nu_{i\,j})$; $i,j = 1,2,\ldots,n$.

The first term of that series was obtained in the form

$$W(+) = \frac{1}{2\pi} \left(\frac{-\psi_o''}{\psi_o} \right)^{2} \quad ,$$

which was taken by Rice [11] as an approximation to the probability function of the interval between successive zero-crossings of a random (noise) process, where

$$\psi_\tau = C(X(t), \; X(t+\tau))$$

is the covariance function of X(t).

The second and third terms in the infinite series involve well known integrals that were discussed by Nabaya [10] and Kamat [5].

The fourth term involves a four-variate integral in the form

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-\sum_{i,j=1}^{4} \nu_{ij}\, x_i\, x_j}\, dx_1\, dx_2\, dx_3\, dx_4$$

where ν_{ij} is the covariance function of the two variables x_i and x_j , $i,j = 1,2,3,4$; $i \neq j$ and $\nu_{ii} = 1$ for $i=1,2,3,4$.

So far no closed form for that integral is given.

In this chapter we reduce this integral to a double integral that is more suitable for numerical computations, in the form

$$\int_0^{\pi/2} d\theta_2 \int_0^{\pi/2} d\theta_3 \; \frac{c_2}{4(1+2\alpha-\beta^2)^{3/2}} \left[\pi/2 - \tan^{-1} \frac{\beta}{\sqrt{1+2\alpha-\beta^2}} - \frac{\beta\sqrt{1+2\alpha-\beta^2}}{1+2\alpha} \right],$$

where

$$\alpha = \alpha(\theta_2,\theta_3) = \nu_{12} c_2^2 c_3 s_3 + \nu_{13}\, c_2 c_3 s_2 + \nu_{23} c_2 s_2 s_3 \quad ,$$

$$\beta = \beta(\theta_2,\theta_3) = \nu_{14}\, c_2 c_3 + \nu_{24}\, c_2 s_3 + \nu_{34}\, s_2$$

and

$$c_i = \cos\theta_i \;, \quad s_i = \sin\theta_i \;, \quad i = 2,3 \,.$$

This double integral is a function of the covariance matrix (ν_{ij}). By taking special values for the covariance functions involved, such as

$$\nu_{12} = -\tfrac{1}{2}\sqrt{\frac{7}{13}} \quad , \quad \nu_{13} = \sqrt{\frac{1}{91}} \quad , \quad \nu_{14} = -\frac{1}{26} \; ,$$

$$\nu_{23} = -\frac{2}{7} \quad , \quad \nu_{24} = \sqrt{\frac{1}{91}} \quad \text{and} \quad \nu_{34} = -\tfrac{1}{2}\sqrt{\frac{7}{13}} \; ;$$

the double integral is evaluated using an IBM 1130 Computer.

Finally in chapter four we investigate some very interesting results of Wong [12] . He considered a random process with a specific covariance function in the form

$$\psi(\tau) = \frac{3}{2}\, e^{-|\tau|/\sqrt{3}} \left(1 - \frac{1}{3}\, e^{-2/\sqrt{3}\,|\tau|}\right).$$

He also obtained the probability function of the interval between successive zeros in a closed form.

CHAPTER I

THE DISTRIBUTION OF ZEROS AND THEIR EXPECTED NUMBER FOR A RANDOM PROCESS

Introduction :

In what follows we discuss some of the results obtained by Rice [11], then in the following chapters we point out the relationships between Rice's results and those of Longuet-Higgins [6], Wong [12], McFadden [7] and Kac [4]. We shall show that the probability distribution of the noise current denoted by $X(t)$, is normally distributed with mean zero and standard deviation $\sqrt{\psi_0}$, where ψ_τ is the correlation function of $X(t)$. Also we shall obtain the joint distribution function of $X(t)$ and $X(t+\tau)$.

In section two of this chapter we obtained the probability that $X(t)$ will have a zero in the interval (t_1, t_1+dt) and consequently the expected number of zeros in the interval (t_1, t_2) is also obtained.

In section three we obtained the conditional probability that the noise current $X(t)$ will pass through zero in the interval $(\tau, \tau+d\tau)$ with a negative slope, given that it passes through zero at $\tau = 0$ with a positive slope. Also we examined the behaviour of this conditional probability, for the region of small spacings between the zeros.

I-1) <u>The distribution of the noise current X(t)</u> :

Rice [11] considered the random noise, which arises from the shot effect in vacuum tubes or from the thermal agitation of electrons in resistors.
In what follows we are concerned with the probability distribution of X(t) and the joint distribution function of X(t) and X(t+ τ).

The noise current can be represented as

$$X(t) = \sum_{n=1}^{N} (a_n \cos \omega_n t + b_n \sin \omega_n t),$$

where $\omega_n = 2\pi f_n$, $f_n = n(\Delta f)$ (I-1.1),

a_n and b_n are taken to be independent random variables which are normally distributed with zero mean and standard deviation $\sqrt{\omega(f_n)(\Delta f)}$. $\omega(f)$ is the power spectrum of X(t). Since a_n and b_n are normally distributed, so are $a_n \cos \omega_n t$ and $b_n \sin \omega_n t$ when t is regarded as fixed. Thus X(t) is the sum of 2N independent normal varieties and consequently is itself normally distributed.

The expected value of X(t) is zero, since $E(a_n) = E(b_n) = 0$,

i.e. $E(X(t)) = 0$.

The mean square value of X(t) is :

$$E(X^2(t)) = \sum_{n=1}^{N} E(a_n^2) \cos^2 \omega_n t + E(b_n^2) \sin^2 \omega_n t$$

$$= \sum_{n=1}^{N} \omega(f_n)(\Delta f) \longrightarrow \int_0^\infty \omega(f)\, df = \psi_0$$