# ON OPTIMAL SOLUTIONS FOR DIFFERENTIAL GAMES

Thesis Submitted For The Degree of

Doctor of Philosophy

In Engineering Mathematics

Physics and Engineering Mathematics Department

رسالة

Submitted By

Eng. Medhat El-Gharib Mohamed Ahmed Metawei



Supervised By

Prof.Dr. Mahmoud Ibrahim Hossam El-Din
Vice dean for Post Graduate Studies & Researches, 28/3/93
Head of Physics and Engineering Mathematics Department,
Faculty of Engineering, Tanta University.

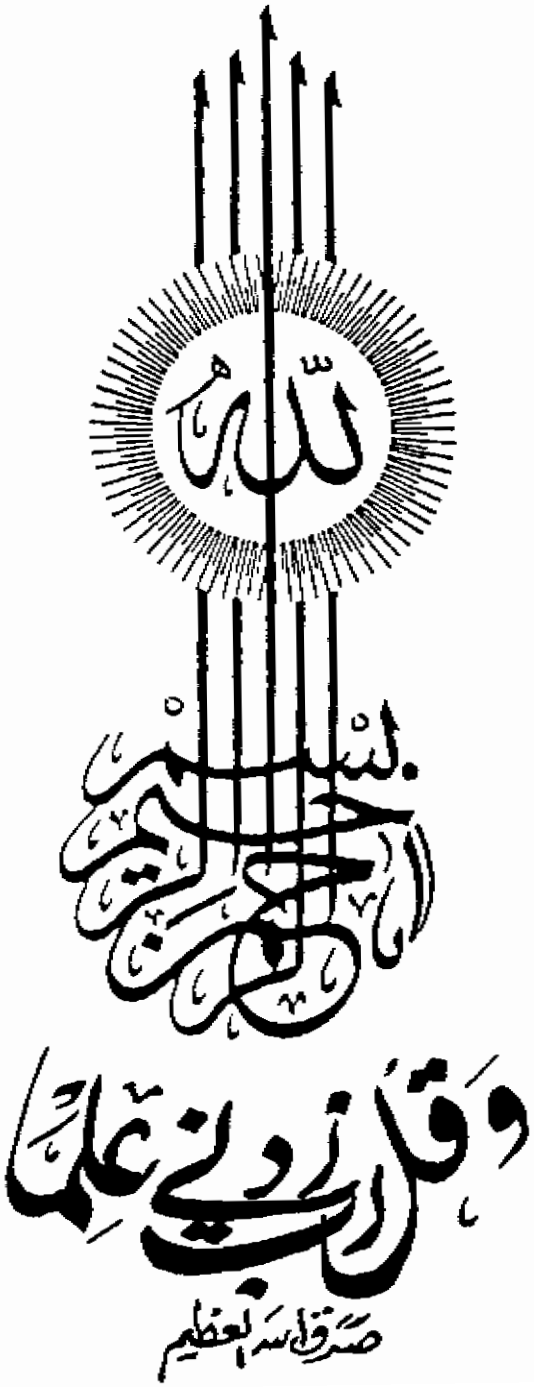
Prof.Dr. Mohamed Ibrahim Hassan
Professor of Mathematics. Ain-Shams University

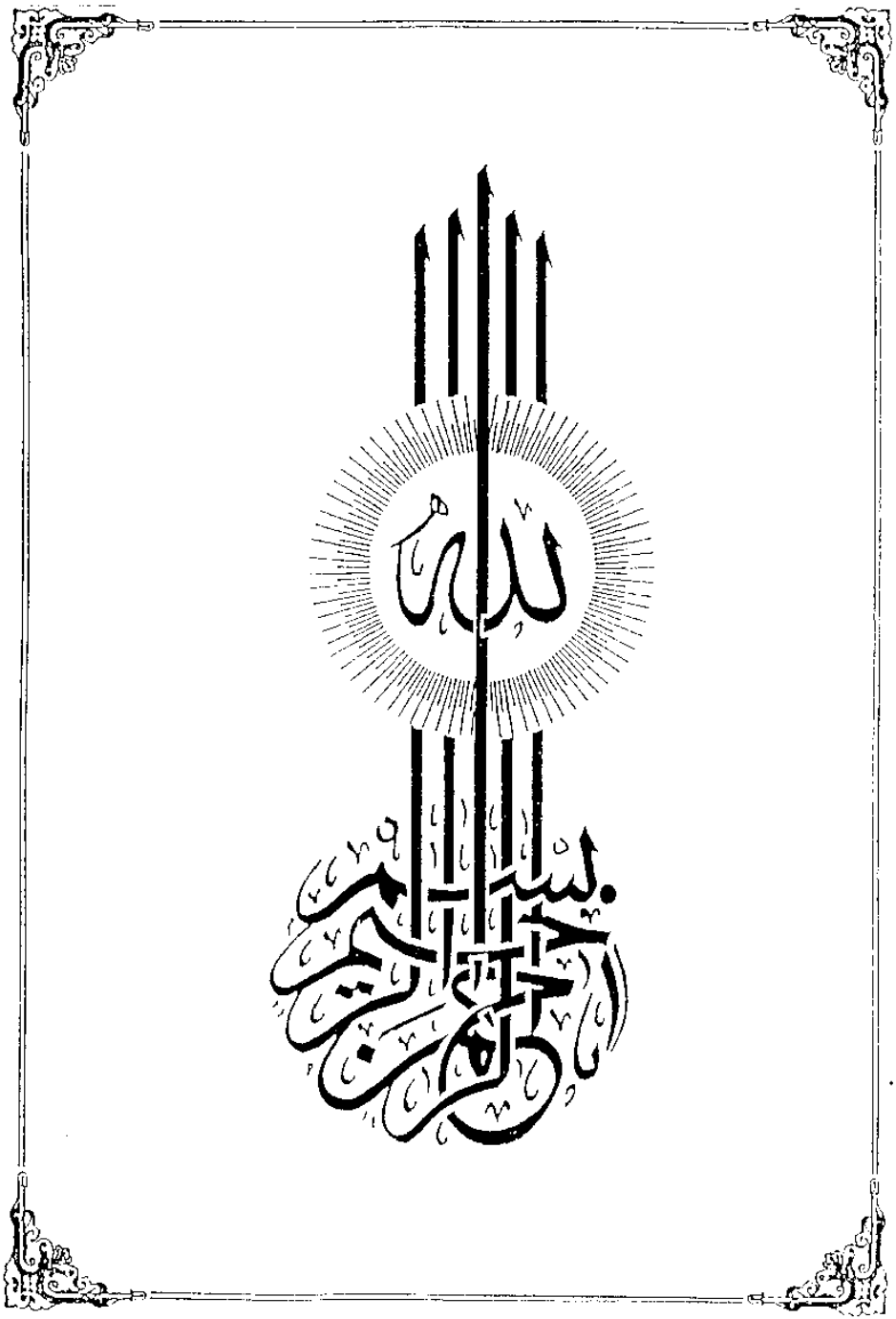
Dr. Abdel-Rehim Abdel-Hamid Abdel-Rehim
Ass.Professor of Mathematics. M.T.C. College.

Faculty of Engineering

Ain-Shams University

1993





# EXAMINERS COMMITTEE

Signature

1- Prof. Dr. Edward Mekhail Ibrahim (Examiner)

Professor of Mathematics, Faculty of Engineering,
Ain-Shams University.

2- Prof. Dr. Yousef Zaki Buttros (Examiner)

Professor of Mathematics, Faculty of Engineering,
Alexandria University.

3- Prof. Dr. Mahmoud Ibrahim Hossam El-Din
(Exam. & Supervisor) 28/3/93

Vice dean for Post Graduate Studies & Researches,
Head of Enineering Mathematics and Physics Department,
Faculty of Engineering, Tnta University.

Date : / / 1993

# CURRICULUM VITAE FOR THE APPLICANT

NAME : MEDHAT EL-GHARIB MOHAMED AHMED METAWEH

BIRTH DATE : 29 NOVEMBER 1950.

EDUCATION :

1- B.Sc. Mech. Eng., Military Technical College (MTC), Cairo, 1973. (V.Good).
2- Diploma in Operations Research, MTC, Cairo, 1982. (V.Good).
3- Diploma in Computer Science, Ain-Shams University, Cairo, 1985. (V.Good).
4- M.Sc. in Operations Research, MTC, Cairo, 1988.

Post Graduate Courses of the M.Sc. Degree :

(1) Mathematics I :
  - Numerical Analysis.
  - Differential Equations.
(2) Mathematics II:
  - Probability & Statistics.
  - Matrix Analysis.
(3) Data Structure & Database.
(4) Advanced Linear Programming.
(5) Advanced Non-Linear Programming.
(6) Stochastic Optimization.
(7) Systems Analysis & Design.
(8) Differential & Difference Equations.

Title of the M.Sc. Thesis :

ON DIFFERENTIAL GAMES WITH GROUND WAR GAMES APPLICATIONS

Medhat

TO : MY MOTHER;

MY WIFE;

AHMED;

BASEM.

# ACKNOWLEDGEMENT

I would like to express my gratitude to my thesis supervisor Professor Dr. Mahmoud Ibrahim Hossam El-Din, Vice dean for Post Graduate Studies and Researches - Faculty of Engineering, Tanta University, for suggesting the problem, his most valuable advice and guidance during elaborating this work. I would also like to thank Professor Dr. Mohamed Ibrahim Hassan, Professor of Mathematics, Faculty of Engineering, Ain-Shams University, and Dr. Abdel-Rehim Abdel-Hamid, Ass. Professor of Mathematics, M.T.C. College, for their encouragement throughout this work.

The demanding nature of any thesis work is hard on the applicant accomplishing the thesis, but it is as equally hard on the applicant's wife . I am deeply indebted to my wife for her great encouragement and sacrifice. I wish to thank her and our sons Ahmed and Basem for their tolerance and understanding throughout the last four years during which I was preparing this work.

# CONTENTS

CHAPTER TWO : MAIN RESULTS

## 2.1 DIFFERENTIAL GAME ANALYSIS FOR OPTIMAL FIRE-SUPPORT STRATEGIES

# ABSTRACT

The thesis discusses the optimal solution for some important mathematical models for differential games. The optimal fire support strategies have been obtained and proved that it depends on the time - course of the combat, the transition surfaces in this case have been obtained. A modefied model for the advertising strategy of two competitive companies has been constructed, discussed, and solved. The results obtained have been published in 3 research papers.

# SUMMARY

Historically, early works in differential games and optimal control theory appeared almost simultaneously, independently of each other. The theory of differential games, however, got a big post immediately after the establishment of optimal control theory in the late fifties. This succession, at first glance, seems to be natural, since a differential game may be viewed upon as a control process where the controls are divided among various players who are willing to use them for objectives with possibly conflict with each other.

A closer look, however, will reveal that the development of the two fields following different lines. Both have the dynamic aspect in common, but differential games have in addition a game-theoretic aspect. The latter overshadows so much the dynamics that the techniques developed in optimal control theory cannot simply be used in differential games. This basic difference is the reason why the picture of the present situation in differential games appears to the casual observer rather unfavorable compared with control theory, and he feels disturbed by the fact that in differential games, mathematical rigour and geometric intuition have still not blended into a satisfactory theory.

This impression has two negative effects : First, it discourages workers in applied science, who could use differential games, to make an effort in learning the background. Secondly, the subject is not adequately represented in the curriculum of many universities and institutes of technology, hence young

engineers and mathematicians rarely have the opportunity to acquire even a modest basic knowledge of the subject.

Our thesis is divided mainly into three parts, the first part is PRELIMINARIES, in which we review the main concepts and definitions of functionals, calculus of variations, optimal control, corner conditions in optimal control, singular solutions in control problems and differential games from the optimal control point of view.

In the second part, chapter one, we discussed the Differential Games Theory and Techniques; showing the different types of Singular Surfaces, then, an overview on the optimal control of a lanchester-type attrition process for a deterministic and a stochastic formulation is done.

Part three, chapter two, is the main part of the thesis, it contains our main results. It is divided into two sections, the first is devoted to the analysis of optimal fire-support strategies using mathematical differential game models. The second section is devoted to analysing the advertising problem as a mathematical differential game.

The problem of the optimal fire-support strategies is one of the most important problems in military operations research. H.K.Weiss discussed the problem in his paper [72] by constructing a differential system describing the model, solving it and using matrix game technique to get the optimal fire-support strategies hence the value of the game. This has imposed many restrections on the objective functions, the loss of possible

and natural time dependence of the strategies. We have studied the problem using differential game theory for different models and several objective functions we got

- Relations between the value of the game and combat duration,
- Dependence of strategies on the time variable and U-singular subarcs,
- Synthesis of extremal strategic variables and some other relations.

Next, a logarithmic excess-advertising model of a duopoly is presented, and Nash optimal open-loop advertising strategies are determined. It turns out that if the two firms use different discount rates, then the optimal strategies will be exponentially decreasing. However, in this case the state equation has no nice solution and must be solved by numerical methods. When both firms use the same discount rate, then the state equation has a simple solution. This solution is also valid for the case when no discounting is performed. Furthermore, when no discounting is performed, the optimal strategies will be simple time-linear decreasing strategies. Finally, we studied how the optimal strategies and trajectories depend on the parameters of the model.

Finally, conclusions and suggested recommendations are presented.

# LIST OF FIGURES AND TABLES

Figures :



Tables :