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BY

ASHRAF FOUAD MOHAMED AHMED

DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
AIN SHAMS UNIVERSITY

SUPERVISORS

Prof. Dr. EL-SAYED M. EL-GHAZZY
FACULTY OF EDUCATION
AIN SHAMS UNIVERSITY

Dr. ABD EL-HAMEED A. M. AMMAR
FACULTY OF SCIENCE
CAIRO UNIVERSITY
BENI-SUEF BRANCH

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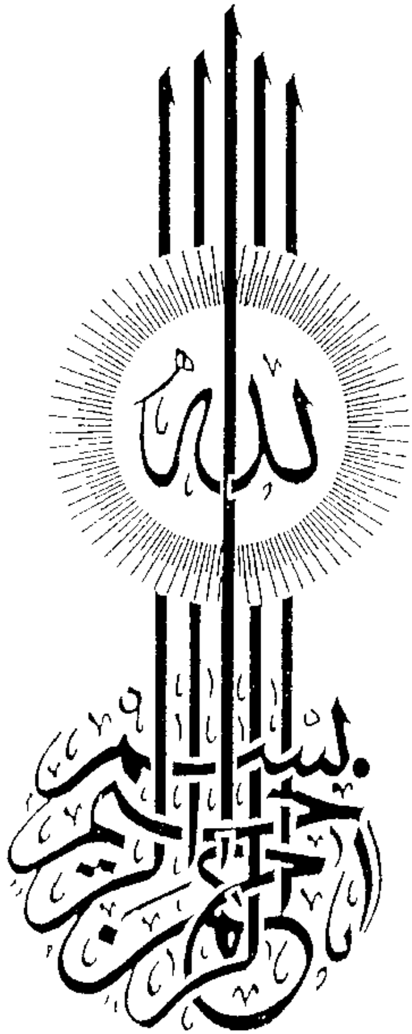
الله

بسم الله الرحمن الرحيم

قالوا سبحانك لا علم لنا إلا ما
علمتنا إنك أنت العليم الحكيم .

صدق الله العظيم
البقرة - ٣٢ -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INTRODUCTION

In order to make a real phenomenon amenable to mathematical description, the mathematical formulation of numerous physical problem results in differential equations which are actually non-linear. In many cases it is possible to replace such a non-linear equations by a related linear equations which approximates the actual non-linear equation closely enough to give useful results. If a phenomenon is described by the following system of differential equations

$$\frac{dy_i}{dt} = \phi_i(t,\ y_1,\ y_2, \dots,\ y_n) \qquad , \qquad i = 1,2,\dots,n.$$

with the initial conditions $y_i(t_o) = y_{io}$, $i = 1,2,\dots,n$. Which are ordinarily the results of measurements and, hence, are inevitably obtained with a certain error.

The question naturally arises as to the effect of small changes in the initial values on the desired solution.

If it turns out that arbitrarily small changes in the initial data are capable of producing a substantial change in the solution, then the solution defined by the chosen initial data is ordinarily devoid of any practical meaning and cannot describe the given phenomenon even approximately.

This brings us to the important question of finding the conditions under which a sufficiently small change in the initial values brings about an arbitrarily small change in the solution.

If t varies on a finite interval $t_o \le t \le T$, the answer to this question is given by the theorem on the continuous dependence of solutions on the initial values.

But if t can take on arbitrarily large values, then the problem is dealt with by the theory of stability.

The subject of stability can be approached from two view points :-

1) A reasonable mathematical problem is to generalized or extend some of the results we obtained concerning the orbits of two-dimensional linear homogeneous system to orbits of non-linear system of dimensional $n > 2$.

2) For more interesting and important is the approach to stability theory from the view point of applications to problems in the physical word, if we assume that some physical system is described with a fair degree accuracy by a system of ordinary differential equations, then how are the solutions of the system of ordinary differential equations reflected in the actual behavior of the physical system?

In recent years, there has been high interest in stability and oscillating phenomena which play a role in regulating biological organisms and in biochemical and chemical reactions. For mathematicians this interest has been partially sparked by new demonstrations of stability and oscillations in predator-prey models.

SUMMARY

This thesis conatins three chapters. The first chapter consists of two parts, the first part is a review of some definitions and some theorems and the second part is a review of some recent results on predator-prey models.

In chapter II, by constructing appropriate Liapunov functionals, asymptotic behaviour of solution of various delay differential systems describing two preys-one predator, competition and symbiosis models has been studied. It has been shown that equilibrium states of these models are globally stable, provided certain conditions in terms of instantaneous and delay interaction coefficients are statisfied.

Starting from this chapter, all the work are new and have been accepted in J. of Appl. Math. and Computations (U.S.A) .

In chpater III, we concerned with the question of effects of dispersal on the linear and non-linear stability of the equilibrium state for a system consisting of two predators and a common prey. It is shown that linearly or non-linearly stable equilibrium state remains so with dispersal as well. Also it is shown that the dispersal has a stabilizing effect.

CHAPTER ONE

1.1 Mathematical Knowledge.

Autonomonus System

Stability

Liapunov Function

Some Important Definitions

1.2 Some Recent Results on Predator - Prey Models.

For Global Stability of Prey-Predator Models

For Global Stability of Competition Models

For Global Stability of Symbiosis Models

Prey-Predator System With Functional Response

1.1 MATHEMATICAL KNOWLEDGE

Autonomous System

Definition 1.1.1 : A system of differential equations in which the independent variable does not appear explicitly , that is , a system of the form

$$X' = F(X) \qquad (1.1.1)$$

is called an *autonomous system*. ([12], P.115)

Definition 1.1.2 : Let $X(t)=\left(X_1(t), X_2(t), \ldots, X_n(t) \right)$ a solution of (1.1.1) such that not all the functions $X_1(t)$, $X_2(t), \ldots, X_n(t)$ are constant functions. Let I be the domain of X(t), the underlying point set of the solution , that is , the set of points

$$C = \left\{ x(t) = \left(x_1(t), x_2(t), \ldots, x_n(t) \right) : t \in I \right\}$$

which is a curve in the intuitive sense , is called *an orbit* of (1.1.1). If n=2, the orbit called *a path*. ([12], P.116)

Definition 1.1.3 : IF $x^0 \in D$, where D is some domain, is such that $F(x^0)=0$, then x^0 is called *an equilibrium point* [or *critical point* or *singular point*] of (1.1.1) .
Note that if x^0 is an equilibrium point of (1.1.1) and for all real t , $x(t)= x^0$, then x(t) is a solution of (1.1.1). ([12], P.118)

Definition 1.1.4 : we consider an autonomous system on the form

$$\frac{dx}{dt} = P(x,y)$$

$$\frac{dy}{dt} = Q(x,y) \qquad (1.1.2)$$

where P(x,y) and Q(x,y) have continuous first partial derivatives for all (x,y) ,the plane xy is called phase plane, we shall define a path to be a curve in xy-plane which may be defined parametrically by more than one solution of (1.1.2). ([46], P.542).

Definition 1.1.5 : A critical point (x_0,y_0) of the system (1.1.2) is called *isolated* if there exists a circle

$$(x-x_0)^2+(y-y_0)^2 = r^2$$

about the point (x_0,y_0) such that this point is the only critical point of (1.1.2) within this circle. ([46], P.545).

Stability

Definition 1.1.6 : Assume that (0.0) is an isolated critical point of the system (1.1.2). Let C be a path of (1.1.2); let x =f(t),y=g(t) be a solution of (1.1.2) defining C parametrically

Let $$D(t) = \sqrt{[f(t)]^2 + [g(t)]^2}$$

denote the distance between the critical point (0,0) and the point R : [f(t), g(t)] on C . The critical point (0,0) is called *stable* if for every number $\varepsilon > 0$, there exists a number

$\delta > 0$ such that the following is true :

Every path C for which $D(t_0) < \delta$ for some value t_0 is defined for all $t \geq t_0$ and is such that $D(t) < \varepsilon$ for $t_0 \leq t < \infty$ ([46], P.551).

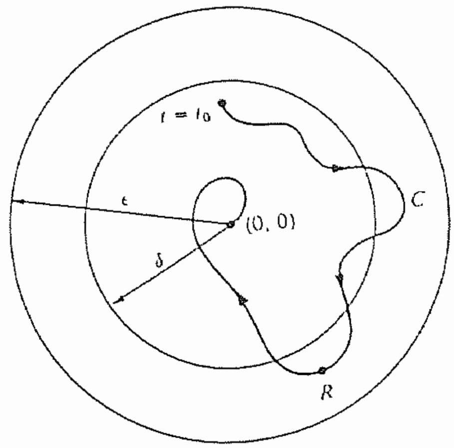

FIGURE (1.1.1)

<u>Definition 1.1.7</u> : Assume that (0,0) is an isolated critical point of the system (1.1.2) . Let C be a path (1.1.2) let $x = f(t)$, $y = g(t)$ be a solution of (1.1.2) representing C parametrically . Let

$$D(t) = \sqrt{[f(t)]^2 + [g(t)]^2}$$

denoted the distance between the critical point (0,0) and the point R : $[f(t), g(t)]$ on C .

The critical point (0,0) is called *asymptotically stable* if

(1) it is stable , and

(2) there exists a number $\delta_0 > 0$ such that if $D(t_0) < \delta_0$ for some t_0, then

$\lim_{t \to \infty} f(t) = 0$, $\lim_{t \to \infty} g(t) = 0$ ([46], P.552).

Definition 1.1.8 : Given the n-dimensional system

$$x' = F(t,x) \qquad (1.1.3)$$

where F has domain D , an open set in (t,x)-space [that is Euclidean (n+1)-space] which includes the positive t-axis and F is continuous on D. Suppose that the solution x(t) of a system (1.1.3) is defined for all $t > \tau$. Then the solution x(t) is *stable* if there exists $t_0 > \tau$ such that $x(t_0) = x^0$ and if x(t) is denoted by $x(t,t_0,x^0)$, the following conditions are satisfied :

(1) There exists a positive constant b such that if

$$|x^1-x^0| < b$$

Then the soultion $x(t,t_0,x^1)$ of a system $x' = F(t,x)$ is defined for all $t \geq t_0$; and

(2) Given $\epsilon > 0$, then there exists $\delta > o$ $\left(\delta = \delta(\varepsilon,F,t_0,x^0)\right)$ such that $\delta \leq b$ and such that if $|x^1-x^0| < \delta$, then for all $t \geq t_0$, $|x(t,t_0,x^1) - x(t,t_0,x^0)| < \varepsilon$. A solution $x(t,t_0,x^0)$ is *asymptotically stable* if :

(1) it is stable; and

(2) there exists $\bar{\delta} > 0$ where $\left(\bar{\delta} = \bar{\delta}(f,t_0,x^0)\right)$ such that $\bar{\delta} < b$ and such that if $|x^1-x^0| < \bar{\delta}$ then

$$\lim_{t\to\infty} |x(t,t_0,x^1)-x(t,t_0,x^0)| = 0. \qquad ([12], P.151).$$

It is clear that , this definition is equivalent to the other definition of stability and asymptotically stable in (1.1.6,1.1.7) and (1.1.8) also we can using the definition :

Given the system

$$\frac{dy_i}{dt} = \phi_i(t, y_1, y_2, \ldots, y_n) \quad i = 1, 2, \ldots, n$$

the solution $\phi_i(t)$ $(i = 1, 2, \ldots, n)$ of this system is called *stable* if for any $\varepsilon > 0$ we choose $\delta(\varepsilon) > 0$ such that for any solution $y_i(t)$ $(i = 1, 2, \ldots, n)$ of the system satisfy the inequlaities $|y_i(t_0) - \phi_i(t_0)| < \delta$ $(i = 1, 2, \ldots, n)$ for all $t \geq t_0$, $|y_i(t) - \phi_i(t)| < \varepsilon$ $(i = 1, 2, \ldots, n)$ hold true and if given an arbitrarily small $\delta > 0$ we have $|y_i(t) - \phi_i(t)| < \varepsilon$ is not true for at least one solution $y_i(t)$ then the solution $\phi_i(t)$ is called *unstable*.

Definition 1.1.9 A critical point is called *unstable* if it is not stable . ([46], P.552).

Example 1.1.1 :

Test for stability the solution of the differential equation

$$\frac{dy}{dt} = -a^2 y, \quad a \neq 0$$

define by the initial condition $y(t_0) = y_0$

the solution $y = y_0 e^{-a^2(t-t_0)}$ is asymptotically stable since

$$\left| y_0 e^{-a^2(t-t_0)} - \bar{y}_0 e^{-a^2(t-t_0)} \right| = e^{-a^2(t-t_0)} |y_0 - \bar{y}_0| < \varepsilon$$

for $t \geq t_0$ if $|y_0 - \bar{y}_0| < \varepsilon \, \bar{e}^{a^2 t_0}$

and $\lim_{t \to \infty} \bar{e}^{a^2(t-t_0)} |y_0 - \bar{y}_0| = 0$