ON DIFFERENTIAL TOPOLOGY

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بسم الله الرحمن الرحيم

M.Sc. COURSES

STUDIED BY THE AUTHOR (FEB. 1977 - FEB.1978)

(AT AIN SHAMS UNIVERSITY)

(i) Functional analysis

3 hours weekly for two semesters.

(ii) Algebraic and differential topology

4 hours weekly for one semester.

(iii) Ordinary differential equations

3 hours weekly for one semester.

(iv) Theory of functions of matrices

3 hours weekly for one semester.

(v) The algebraic eigenvalue problem

4 hours weekly for one semester.

PREFACE

The thesis consists of three chapters. Chapter I involves very many definitions in the subjects of immersion and imbedding and fibre bundles. It involves the statements of two theorems which are needed for the material in chapter II.

Chapter II considers two important topics. The first is the proof of the so-called the"Easy-Whitney Imbeeding Theorem". The second topic is the proof of Epestien-Schwarzenberger two theorems of 1962, on imbedding.

Chapter III exposes two interesting topics. The first topic is cell complexes and combinatorial equivalence. The second topic is the immersions and imbeddings of complexes.

In the thesis we always consider the complex case which is rather slightly more difficult than the real case.

C O N T E N T S

CHAPTER I

PRELIMINARIES

In this chapter we give some preliminaries, following [2], [3], [5] and [6], which will be required throughout the thesis.

1. SURVEY ON IMBEDDING AND IMMERSION [2] AND [5]

Definition (1.1.1)

A subset $B \subset \mathcal{T}$ is called an open base for the topological space $(X, \mathcal{T})$ if each member of $\mathcal{T}$, the topology for X, is the union of a family of members of B.

Definition (1.1.2)

A subset $A \subset X$ is said to be dense in X if the closure $\bar{A}$ in $(X, \mathcal{T})$ is equal to X.

Definition (1.1.3)

A subset $B \subset X$ is said to be nowhere dense if the interior $\overset{\circ}{B}$ of B is the void set Φ, i.e. B is nowhere dense in X $\Longrightarrow \overset{\circ}{B} = \Phi$

Definition (1.1.4)

An n-cube $K \subset C^n$, where C^n is the complex n-space, of edge $\lambda > o$ is a product of closed intervals of length λ ;

$$K = I_1 x \dots x \; I_n \subset C \underbrace{x \dots x}_{\text{n-copies}} C = C^n$$

thus

$$I_j = \left[a_j , a_j + \lambda \right] \subset C .$$

The measure (or n-measure) of K is

$$\mu(K) = \mu_n(K) = \lambda^n .$$

Definition (1.1.5)

A subset $X \subset C^n$ has measure zero if for every $\varepsilon > o$ it can be covered by a family of n-cubes , the sum of whose measures is less than ε .

Definition (1.1.6)

(a) A family $\{ A_i \}_{i \in I}$ of open subsets is called open covering of X if and only if

$$X = \bigcup_{i \in I} A_i$$

(b) A family $\{ D_i \}_{i \in I}$ of closed subsets is called closed covering of X if and only if

$$X = \bigcup_{i \in I} D_i$$

Definition (1.1.7)

The pair (U,V) of the space X is called a partition of X if and only if

$$A \neq \phi, \qquad B \neq \phi ,$$

$$A \cap \bar{B} = \phi \quad , \quad B \cap \bar{A} = \phi, \text{ and}$$
$$X = A \cup B.$$

Definition (1.1.8)

The space (X, τ) is disconnected if and only if X has a partition (U , V) and the sets U and V form a seperation of X.

Definition (1.1.9)

The space (X , τ) is connected if and only if it is not disconnected. A subset $A \subset X$ is connected if and only if the space (A, τ_A) is connected, where
$\tau_A = \tau | A$

Definition (1.1.10)

The topological space (X , τ) is said to be Hausdorff space if for every z_1 and $z_2 \in X$ and $z_1 \neq z_2 \Rightarrow$ there exist $A \in U(z_1)$ and $B \in U(z_2)$ such that $A \cap B = \phi$, where $U(z_1)$ and $U(z_2)$ are , respectively , the neighbourhoods of z_1 and z_2

Definition (1.1.11)

The space X is locally connected at the point $z \in X$ if and only if given any open set $U \subset X$ containing z there exists an open connected subset $V \subset X$ such that $z \in V \subset U$. If X is locally connected at each point $z \in X$, then X is called <u>locally connected space</u>. A subset $A \subset X$ is locally connected if and only if the space (A, τ_A) is locally connected .

<u>Definition (1.1.12)</u>

Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of subsets of the space X and $B \subset X$, the family $\{A_\alpha\}_{\alpha \in \Lambda}$ <u>cover</u> B if $B \subset \bigcup_{\alpha \in \Lambda} A_\alpha$. If Λ is finite and $\{A_\alpha\}_{\alpha \in \Lambda}$ cover B, then $\{A_\alpha\}_{\alpha \in \Lambda}$ is called <u>finite cover</u> of B. If each A_α, $\alpha \in \Lambda$ is open (closed) in X and $\{A_\alpha\}_{\alpha \in \Lambda}$ covers B, then $\{A_\alpha\}_{\alpha \in \Lambda}$ is called an <u>open (closed) cover</u> of B.

<u>Definition (1.1.13)</u>

Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a cover of $B \subset X$. Then the family $\{A_\beta\}_{\beta \in \Omega \subset \Lambda}$ is a <u>subcover</u> of $\{A_\alpha\}_{\alpha \in \Lambda}$ for B if $\{A_\beta\}_{\beta \in \Omega \subset \Lambda}$ is a cover of B.

<u>Definition (1.1.14)</u>

A space X is called <u>compact</u> if each open cover of X has a finite subcover for X. A subset A of the space (X , τ) is compact if the space (A , τ_A) is compact.

<u>Definition (1.1.15)</u>

A topological space is said to be <u>locally compact</u> if each of its points has a compact neighbourhood.

Definition (1.1.16)

Let (X , τ) and (Y , τ') be two topological spaces and $f : X \longrightarrow Y$ be a map . The function f is continuous at the point $z \in X$ if given any open set $V \subset Y$ containing $f(z)$, there exists an open set $U \subset X$ containing z such that $f(U) \subset V$.

Definition (1.1.17)

Let $f : X \longrightarrow Y$ be a bijective function from the space X to the space Y. If f is open and continuous , then f is called a homeomorphism. If f is a homeomorphism from X to Y then the spaces X and Y are said to be homeomorphic , denoted by $X \cong Y$.

Definition (1.1.18)

Any topological space X is called n-dimensional complex locally space if every point $z \in X$ has a neighbourhood which is homeomorphic to an open subset of C^n or every point on X is homeomorphic to an n-cell in C^n.

Definition (1.1.19)

A chart on X is a homeomorphism of an open subset of X onto an open subset of C^n .

Definition (1.1.20)

A chart on X defines a set of n-coordinate complex functions on any open subset of X got by decomposition of a chart with the various complex projection mappings $C^n \longrightarrow C^1$; such an open subset of X with the set of such coordinate functions is known as the coordinate neighbourhood.

Definition (1.1.21)

An atlas on a complex space X is a collection of charts on X whose domain covers X .

Definition (1.1.22)

A function f is called C^{ω}-function if it is analytic i.e. it satisfies the Cauchy - Riemann equations(the C - R - equations) .

Definition (1.1.23)

A C^{ω}-n- manifold X^n is defined as paracompact Hausdorff-space with a family $\mathfrak{F}$ defined on X , $\mathfrak{F}$ is the set of all continuous complex functions on X such that

(a) $\mathfrak{F}$ is local

(b) $\mathfrak{F}$ is analytically closed , and

(c) $(X, \mathfrak{F})$ is locally Euclidean in the complex sense.

Definition (1.1.24)

Let M_1 and M_2 be two C^{ω}-manifolds with different dimensions , a C^{ω}- function $f : M_1 \longrightarrow M_2$ is a C^{ω}- imbedding if

(a) the function f is homeomorphic onto, and

(b) the rank of f equal to the dimension of M_1 at all points of M_1.

Definition (1.1.25)

A C^{ω}- map f of M_1 to M_2 is called an immersion if f has rank equal to the dimension of M_1.

Definition (1.1.26)

A complex manifold is a Hausdorff space covered by neighbourhoods each homeomorphic to a cell in n-dimensional complex space C^n , such that when two neighbourhoods overlap the local coordinates transform by a complex analytic transformation.

Definition (1.1.27)

An analytic submanifold $N_q \subset M_m$ is a closed subset such that if $z \in N$ there is a neighbourhood U of z , in with local coordinates $W^1, \ldots, W^m$, such that $N \cap U$ is the set of points in M at which $W^{q+1}, \ldots, W^m$ all vanish.

Definition (1.1.28)

Any complex form can be written in the form $\omega = \alpha + i\beta$ where α, β are real forms. Then $\bar{\omega} = \alpha - i\beta$. We define

$$dz^k = dx^k + idy^k \ , \ d\bar{z}^k = dx^k - idy^k .$$

Then since dx^k , dy^k form a base for the space of complex C^ω-forms , dz^k, $d\bar{z}^k$ form a base also . Then a form of degree "r" can be written in the form

$$\omega = \omega_{r,o} + \omega_{r-1,1} + \cdots + \omega_{o,r} ,$$

where $\omega_{p,q}$ has degree p in dz^k and degree q in $d\bar{z}^k$.

Definition (1.1.29)

We define the (p,q) -projection $\pi_{p,q}$ by $\pi_{p,q}\,\omega = \omega_{p,q}$. Then $\pi_{p,q}$ does not depend on the choice of the coordinate system because if $W^1, \ldots, W^m$ is another system of local complex coordinates, the C - R equations show that $d\omega^j$ is a linear combination of dz^k , and $d\bar{\omega}^j$ is a linear combination of $d\bar{z}^k$. Therefore, the complex C^ω-forms form a bigraded ring over the complex nubmbers. The form $\omega_{p,q}$ is said to have bidegree (p,q) . Hence,

$$\omega_{p,q} = \frac{1}{p!q!} \sum a_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q},$$

where $a_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q}$ is antisymmetric in the i's and in the j's and is a complex valued C^{∞}- function of the real coordinates.

Then $d\omega_{p,q} = \partial\omega_{p,q} + \bar{\partial}\omega_{p,q}$ where $\partial\omega_{p,q}$ has bidegree (p + 1,q) and $\bar{\partial}\omega_{p,q}$ has bidegree (p,q+1). Therefore, $d = \partial + \bar{\partial}$ where ∂ has bidegree (1,o) and $\bar{\partial}$ has bidegree (o,1). For general forms we define

$$\partial = \sum_{p,q} \pi_{p+1,q} \, d \, \pi_{p,q} \quad \text{and} \quad \bar{\partial} = \sum_{p,q} \pi_{p,q+1} \, d \, \pi_{p,q} .$$

Then $d = \partial + \bar{\partial}$ in general. It follows that

$$o = d^2 = \partial^2 + \bar{\partial}^2 + \partial\bar{\partial} + \bar{\partial}\partial$$

so from degree considerations we get that

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = o.$$

Definition (1.1.30)

A complex valued function f(x,y) is holomorphic if and only if $\bar{\partial} f = o$. Hence, we define a holomorphic form to be a form ω of type (p,o) such that $\bar{\partial}\omega = o$. This is equivalent to saying that the coefficients of ω are holomorphic functions.