ON THE STEENRODS HOMOLOGY THEORY

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PREFACE.

The homology theory is the oldest and most extensively developed portion of algebraic topology, and may be regarded as the main body of the subject, its axiomatic characterization has been given firstly by S. Eilenberg and N. Steenrod [7]. Roughly speaking, a homology theory assigns groups to topological spaces, and homomorphisms to continuous maps of one space into another. One of the most important homology theories is that due to N.Steenrod [12], who had introduced in 1940 new homology groups for compact metric spaces.

Many mathematicians have investigated and generalized Steenrod's homology groups; we mention among them, Milnor [11], Citnikov [3], Borel and Moore [1], Chogoshvili [2] and A.A.Dabbour [4,5].

The present thesis consists of three chapters:

The first chapter contains basic ideas from topology, algebra and algebraic topology, namly the category of simplicial complexes which is necessary to give any homological structure.

The second chapter seems to be original; it deals with a slight generalization of the concept of direct spectrum to include the case when there may be more than one homomorphism between two groups of the spectrum.

The first article of the third chapter discusses the Steenrod's homology groups of the regular cycles of compact metric spaces, and we give a complete proof for the homotopy axiom for these groups.

The second article of chapter (III) seems to be original; it deals with the projective homology group, which is one of the generalizations of the Steenrod's homology groups. This generalization has some points of contiguity with the theory of double complexes, to know to what extent the Steenrod's homology groups can be constructed by using some ideas of homological algebra. It is proved that the projective homology construction is a ∂-functor from the category of compact metric spaces and continuous maps to the category of abelian groups and homomorphisms.

CONTENTS

CHAPTER I

SOME BASIC IDEAS

In this chapter we introduce a number of definitions and conventions which we need in our thesis.

1. Algebraic and Topological Concepts

Definition 1.1.1. A lower sequence of groups is a collection $G = \{ G_q , \Phi_q \}$, where for each integer q (positive, negative, or zero), G_q is a group and $\Phi_q : G_q \longrightarrow G_{q-1}$ is a homomorphism.

The lower sequence G is said to be exact if, for each integer q, $\operatorname{Im} \Phi_{q+1} = \operatorname{Ker} \Phi_q$ (see [7]), where $\operatorname{Im} \Phi_{q+1} = \Phi_{q+1}(G_{q+1})$ and $\operatorname{Ker} \Phi_q = \{ g \in G_q / \Phi_q(g) = 0 \}$.

Definition 1.1.2. An upper sequence of groups is a collection $G = \{ G^q , \Phi^q \}$, where for each integer q (positive, negative, or zero), G^q is a group and $\Phi^q : G^q \longrightarrow G^{q+1}$ is a homomorphism.

An upper sequence G is exact if, for each integer q, $\operatorname{Im} \Phi^{q-1} = \operatorname{Ker} \Phi^q$.

There is a one-to-one correspondence between the set of all lower sequences and the set of all upper sequences, for, if $G = \{ G_q , \Phi_q \}$ is a lower sequence and we set $G^q = G_{-q}$

and $\Phi^q = \Phi_{-q}$ for each q, then $\{ G^q , \Phi^q \}$ is an upper sequence.

The following definitions are given for lower sequences and the corresponding definitions for upper sequences are to be obtained using the above transformation.

Definition 1.1.3 A lower sequence $G' = \{ G'_q , \Phi'_q \}$ is said to be a subsequence of the lower sequence

$$G = \{ G_q , \Phi_q \} ,$$

written $G' \subset G$, if for each integer q, $G'_q \subset G_q$ and $\Phi'_q = \Phi_q \setminus G'_q$, i.e. Φ'_q is the restriction of the map Φ_q on the subgroup G'_q of G_q.

Definition 1.1.4. If $G = \{ G_q , \Phi_q \}$, $G' = \{ G'_q , \Phi'_q \}$ are two lower sequences, a homomorphism $\Psi : G \longrightarrow G'$ is a sequence of homomorphisms $\{ \Psi_q \}$ such that, for each integer q, $\Psi_q : G_q \longrightarrow G'_q$ and the following commutativity relation hold:

$$\Phi'_q \Psi_q = \Psi_{q-1} \Phi_q .$$

If each Ψ_q is an isomorphism, Ψ is said to be an isomorphism.

Definition 1.1.5. If $L = \{ L_q , \Psi_q \}$ is a subsequence of the lower sequence $G = \{ G_q , \Phi_q \}$, the factor sequence G/L of G by L is the lower sequence composed of the factor groups G_q/L_q and the homomorphisms $\tilde{\Phi}_q : G_q/L_q \to G_{q-1}/L_{q-1}$ which are induced by the Φ_q.

Definition 1.1.6. Let $\{X^\alpha\}$ be a collection of sets indexed by a set M. The product of the collection, denoted by $\prod_{\alpha\in M} X^\alpha$, is the totality of functions $x = \{x^\alpha\}$ defined for every $\alpha \in M$ and such that x^α, the value of x on α, is an element of X^α. In particular, if M consists of the first n natural numbers, 1, 2, ..., n, then the product of the collection, is the set of all n-tuples $(x^1, x^2, \ldots, x^n)$ such that $x^i \in X^i$, $i = 1, 2, \ldots, n$, and will be denoted by $X^1 \times X^2 \times \ldots \times X^n$.

If each X^α is a topological space, a topology is introduced in the product of the collection $\{X^\alpha\}$ as follows: if a finite number of the X^α's are replaced by open subsets $U^\alpha \subset X^\alpha$, the product of the resulting collection is a subset of $\prod_{\alpha\in M} X^\alpha$, and is called a rectangular open set of $\prod_{\alpha\in M} X^\alpha$. Any union of rectangular open sets, is called an open set of the product. The product with this topology, is called the topological product.

If each X^α in the collection $\{X^\alpha\}$ is an abelian group, then an addition is defined in the product $\prod_{\alpha\in M} X^\alpha$ by the usual method of adding functional values:

$$(x + x')^\alpha = x^\alpha + x'^\alpha.$$

In this way the product $\prod_{\alpha \in M} X^{\alpha}$ becomes an abelian group and is called the direct product of the groups $\{ X^{\alpha} \}$.

Definition 1.1.7. Let $\{ G^{\alpha} \}$ be an indexed collection of groups. Their direct sum $\sum_{\alpha \in M} G^{\alpha}$, (M is the set of indices), is the subgroup of their direct product $\prod_{\alpha \in M} G^{\alpha}$ consisting of those elements $x = \{ x^{\alpha} \}$ having all but a finite number of x^{α}, s equal to zero, i.e., $x^{\alpha} = 0 \in G^{\alpha}$ for all but a finite number of $\alpha \in M$.

Definition 1.1.8. A relation "$\prec$" in a set M is called quasi-order if it is reflexive and transitive. A directed set M is a quasi-order set such that for each $\alpha, \beta \in M$, there exists an element $\gamma \in M$ for which $\alpha \prec \gamma$ and $\beta \prec \gamma$.

A subset M' of M is cofinal in M if, for each $\alpha \in M$, there exists an element $\beta \in M'$ such that $\alpha \prec \beta$.

If M and N are directed sets, a map $\Phi : M \dashrightarrow N$ is an ordered-preserving function from M to N, i.e., $\alpha \prec \beta$ in M implies $\Phi(\alpha) \prec \Phi(\beta)$ in N.

2. Categories and Functors

An algebraic representation of a topology is a mapping from the topology to algebra. Such a representation converts a topological problem into an algebraic one to the end that, with sufficiently many representations, the topological problem will be solvable if (and only if) all the corresponding algebraic problems are solvable, (see [7], [13]).

The definition of a representation, formally called a functor, is given in this article. This article is devoted to the concept of category, because functors are functions, with certain natural properties, from one or several categories to another.

Definition 1.2.1 A category may be thought of as consisting of sets, possibly preserving the additional structures. More precisely, a category C consists of

(a) a class of objects,

(b) for every ordered pair of objects X and Y there is a set $hom_C(X,Y)$ of morphisms with domain X and range Y; if $f \in hom_C(X,Y)$ we write $f : X \dashrightarrow Y$,

(c) for every ordered triple of objects X, Y, and Z, a function associating to a pair of morphisms $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ their composite $gf : X \dashrightarrow Z$, satisfies the following two axioms:

Associativity: if $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$, and $h : Z \dashrightarrow W$, then $(h\ g)\ f = h\ (g\ f) : X \dashrightarrow W$

Identity: for every object Y there is a morphism $I_y : Y \dashrightarrow Y$ such that if $f : X \dashrightarrow Y$, then $I_y\ f = f$, and if $h : Y \dashrightarrow Z$, then $h\ I_y = h$.

An example of the category is the category of abelian groups and their homomorphisms.

<u>Definition 1.2.2.</u> Let C_1 and C_2 be categories. A covariant functor T, from C_1 to C_2, consists of an object function, which assigns to every object X of C_1 an object $T(X)$ of C_2, and a morphism function which assigns to every morphism $f : X \dashrightarrow Y$ of C_1 a morphism $T(f) : T(X) \dashrightarrow T(Y)$ of C_2 such that:

(a) $T(I_X) = I_{T(X)}$,

(b) $T(gf) = T(g)\ T(f)$.

We shall have occasion to compare functors with others. This is done by means of a suitably defined map between functors.

<u>Definition 1.2.3:</u> Let T_1 and T_2 be covariant functors from a category C_1 to a category C_2. A natural transformation Φ from T_1 to T_2, is a function from the objects of C_1 to morphisms of C_2 such that for every morphism $f : X \dashrightarrow Y$

of C_1 the following diagram is commutative

$$\begin{tikzcd}
T_1(X) \arrow[r, "T_1(f)"] \arrow[d, "\Phi(X)"'] & T_1(Y) \arrow[d, "\Phi(Y)"] \\
T_2(X) \arrow[r, "T_2(f)"] & T_2(Y)
\end{tikzcd},$$

i.e. $\quad \Phi(Y)\, T_1(f) = T_2(f)\, \Phi(X)$.

<u>Definition 1.2.4.</u> A category with couples (briefly: c-category) is a category C in which certain pairs (α, β) of maps, called couples, are distinguished, subject to the sole condition that the composition $\beta\,\alpha$ is defined. If $\alpha : X \dashrightarrow Y$, $\beta : Y \dashrightarrow Z$ is such couple, we write

$$(\alpha, \beta) : X \dashrightarrow Y \dashrightarrow Z .$$

An example of c-category is that obtained from the category $\mathcal{G}$ of abelian groups and their homomorphisms by defining couples $(\Phi, \Psi) : G_1 \dashrightarrow G_2 \dashrightarrow G_3$, whenever $\Phi : G_1 \dashrightarrow G_2$ has a kernel zero, $\Psi : G_2 \dashrightarrow G_3$ is onto, and $\operatorname{Ker} \Psi = \operatorname{Im} \Phi$.

Now if $(\alpha, \beta) : X \dashrightarrow Y \dashrightarrow Z$

and $\qquad (\alpha_1, \beta_1) : X_1 \dashrightarrow Y_1 \dashrightarrow Z_1$

are couples in a c-category C, a map from the couple (α, β) to the couple (α_1, β_1) consists of three maps $\gamma_1 : X \dashrightarrow X_1$,

$\gamma_2 : Y \dashrightarrow Y_1$ and $\gamma_3 : Z \dashrightarrow Z_1$ in the category C such that commutativity holds in the two squares of the diagram:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\gamma_1"'] & Y \arrow[r, "\beta"] \arrow[d, "\gamma_2"] & Z \arrow[d, "\gamma_3"] \\ X_1 \arrow[r, "\alpha_1"] & Y_1 \arrow[r, "\beta_1"] & Z_1 \end{tikzcd} .$$

<u>Definition 1.2.5:</u> Let C be a c-category. Consider the system $H = \{ H_q(X) , \alpha_* , \partial(\alpha,\beta) \}$, where (1) for each object $X \in C$ and each integer q , $H_q(X)$ is a group, (2) for each map $\alpha : X \dashrightarrow Y$ in C and each integer q, $\alpha_* : H_q(X) \dashrightarrow H_q(Y)$ is a homomorphism, (3) for each couple $(\alpha,\beta) : X \dashrightarrow Y \dashrightarrow Z$ and each integer q, $\partial(\alpha,\beta): H_q(Z) \dashrightarrow H_{q-1}(X)$ is a homomorphism. Such a system H is called a covariant ∂-functor on the c-category C provided the following four axioms hold:

Axiom 1. If α = identity, then α_* = identity.

Axiom 2. $(\beta\ \alpha)_* = \beta_*\ \alpha_*$.

Axiom 3. If γ_1, γ_2, γ_3 form a map of the couple $(\alpha , \beta) : X \dashrightarrow Y \rightarrow Z$ into the couple (α_1,β_1): $X_1 \rightarrow Y_1 \rightarrow Z_1$, then commutativity holds in the diagram

$$\begin{tikzcd} H_q(Z) \arrow[r, "\gamma_{3*}"] \arrow[d, "\partial(\alpha,\beta)"'] & H_q(Z_1) \arrow[d, "\partial(\alpha_1,\beta_1)"] \\ H_{q-1}(X) \arrow[r, "\gamma_{1*}"] & H_{q-1}(X_1) \end{tikzcd} .$$