ON INFINITE MATRICES WITH SOME APPLICATIONS

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(1) Abstract algebra — 2 hours per week
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(2) Functional analysis — 2 hours per week

(3) Algebraic infinite matrices — 2 hours per week

(4) The algebraic eigenvalue problem — 2 hours per week

(5) Linear programming — 2 hours per week

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PREFACE

The thesis consists of three chapters. In chapter I we investigate the order of magnitude of the elements of a matrix function $F(A)$ where A is an algebraic semi block matrix of which the elements are of certain orders of magnitude. In § 1, we mainly summarize the known results in the subject, but we add a few new results; these new results are in theorems 1.4, 1.5 and 1.6, and in corollaries 1.3, 1.4 and 1.5. We give in § 2, our main contribution to the subject; the results in theorems 2.1 to 2.11 and in corollaries 2.1 to 2.17 are new in the subject.

In chapter II, we investigate the order, and type on a circle, of basic sets of polynomials associated with non-singular matrix functions $F(F)$ where F is an algebraic semi block matrix of which the elements are of certain orders of magnitude. In § 1, we summarize the known results in the subject, and in § 2 we give our own contribution. The results in theorems 2.2 to 2.[illegible], in corollaries 2.1 to 2.[illegible] and in corollaries 2.[illegible] to 2.17 are entirely new in the subject. The results in theorems 2.1, corollary 2.7 and theorem 2.10 are respectively improvements of the (known) results in theorems 1.4, 1.3 and 1.2 of § 1. The result in theorem 2.11 involves that in theorem 1.5 of § 1.

In chapter III, we investigate two problems. The first is the order of magnitude ofthe lengths of the rows of a matrix function F(A) where A is an algebraic row-finite matrix of which the lengths of the rows are of given orders of magnitude. The second problem is the order of magnitude of the elements of F(A) when A is an algebraic row-finite matrix of which the lengths of the rows and the elements are of given orders of magnitude. The first problem is treated in § 1, in which we summarize the known results and give a few additions. The second problem is treated in § 2, in which we first summarize the known results and then give our main contribution to the subject, this being in theorem 2.5 an d in corollaries 2.7 to 2.13.

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CHAPTER I

ON THE ORDER OF MAGNITUDE OF THE ELEMENTS OF A MATRIX FUNCTION F(A) WHERE A IS AN ALGEBRAIC SEMI BLOCK MATRIX

1. Introduction, and some results on the order of magnitude of the elements of F(A)

An infinite matrix A is said to be algebraic if it is self-associative and satisfies a polynomial equation of the form

$$(1.1)\qquad f(A) = A^m + k_1 A^{m-1} + k_2 A^{m-2} + \dots + k_m I = 0$$

where I is the infinite unit matrix. If this equation is the equation of least degree satisfied by the matrix A, A is said to be algebraic of degree m.

An algebraic infinite matrix A shares square matrices many of their properties. If the minimum equation of A is

$$(1.2)\qquad f(A) \equiv \prod_{i=1}^{t} (A - u_i I)^{m_i} = 0, \qquad \sum_{i=1}^{t} m_i = m$$

and if

$$(1.3)\qquad \frac{1}{f(z)} = \sum_{i=1}^{t} \frac{p_i(z)}{(z - u_i)^{m_i}} ,$$

$$(1.4)\qquad f_i(z) = p_i(z) \; f(z)/(z - u_i)^{m_i}$$

then for a function F(z) which is analytic in a domain D in which the "scalar roots" $u_1, u_2, \dots, u_t$ of A all lie, we have the polynomial representation for $F(A)$

$$(1.5)\quad F(A) = \sum_{i=1}^{t} Z_i(A) \left\{ \sum_{j=0}^{m_i-1} (A - u_i I)^j D_i^j \;/ j! \right\} F(u_i)$$

where $D_i = d/du_i$ [10].

The polynomial representation in (1.5) can be reduced by the use of the minimum equation to a polynomial of degree at most m-1 in A. Indeed we have the polynomial representation [10],

$$(1.6)\quad F(A) = \alpha_1 A^{m-1} + \alpha_2 A^{m-2} + \dots + \alpha_m I$$

where

$$(1.7)\quad \alpha_r = \frac{1}{2\pi i} \int_C \left(z^{r-1} + k_1 z^{r-2} + k_2 z^{r-3} + \dots + k_{r-1} \right) \frac{F(z)}{f(z)}\, dz,$$

$r = 1, 2, \dots, m$, the integrals being taken round a contour C lying in the domain in which $F(z)$ is analytic and enclosing the spectrum of A (i.e., the poles of $1/f(z)$) .

When $F(z)$ is analytic in a domain D within which the scalar roots u_i, $i = 1, 2, \dots, t$, of an algebraic infinite matrix A all lie, and $|h|$ is small enough so that $u_i + h$, $i = 1, 2, \dots, t$ all lie within D, then

$$\lim_{|h| \to 0} \left[\left\{ F(A + hI) - F(A) \right\} / h \right]$$

exists and is equal to

$$\sum_{i=1}^{t} f_i(A) \left\{ \sum_{j=0}^{m_i-1} (A - u_i I)^j D_i^{j+1} \right\} F(u_i).$$

This limit is called the (first) derivative of F(A) with respect to the matrix A. According to this definition, F(A) has derivatives of all orders with respect to A, the n-th derivative being

$$\sum_{i=1}^{t} f_i(A) \left\{ \sum_{j=0}^{m_i-1} (A - u_i I)^j D_i^{j+n} \right\} F(u_i).$$

Therefor , F(A) is said to be an analytic function of A.

An infinite matrix of the form

$$(1.8) \qquad A = \begin{bmatrix} A_{11} & 0 & 0 & \cdots \\ A_{21} & A_{22} & 0 & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where A_{11}, A_{22}, A_{33}, ... are square submatrices of orders, say r_1, r_2, r_3, ... is called a (lower) semi block matrix. A semi block matrix B is said to be of the same structure type as the semi block matrix A, in (1.8), if it can be written in the form

$$(1.9) \qquad B = \begin{bmatrix} B_{11} & 0 & 0 & \cdots \\ B_{21} & B_{22} & 0 & \cdots \\ B_{31} & B_{32} & B_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where every submatrix B_{ij} is of the same order as that of the corresponding submatrix A_{ij} of A. Evidently then B is a semi block matrix of the same structure type as a given semi block matrix, the matrices $A + B$ and AB are semi block matrices of the same structure type as A. Also a scalar matrix may be considered as a semi block matrix of any required structure type. Thus semi block matrices having the same structure type as a given semi block matrix A form a field (in the sense of the definition of a field of infinite matrices where an element of the field is not required to have an inverse element in the field). This field is called the field of the semi block matrix A. Since an analytic function $F(A)$ of an algebraic infinite matrix A is represented by a polynomial in A, every analytic function $F(A)$ of an algebraic semi block matrix A lies in the field of A.

It is our purpose in this chapter to investigate the order of magnitude of the elements f_{ij} of $F(A)$ where A is an algebraic semi block matrix of which the elements a_{ij} are of some given order of magnitude.

Given a semi block matrix $A = \{a_{ij}\}$, $i, j = 1, 2, 3, \ldots$, for any (row index) i, there is a number $v(i)$ such that

(1.10) $r_1 + r_2 + \ldots + r_{v(i)-1} < i \leq r_1 + r_2 + \ldots + r_{v(i)}$.

We usually write

(1.11) $r_1 + r_2 + \ldots + r_{v(i)-1} = t(i)$,

(1.12) $r_1 + r_2 + \ldots + r_{v(i)} = s(i)$.

Then

(1.13) $\quad t(i) < i \leq s(i)$.

A mal S. Mahmoud [9] has proved the following results:

<u>Theorem 1.1</u>: Let $A = \left[a_{ij} \right]$ be an algebraic semi block matrix satisfying

(1.14) $$\overline{\lim_{i \to \infty}} \frac{\log \max_j |a_{ij}|}{i^u} = a, \quad 0 < u < \infty ,$$

$0 < a < \infty$,

(1.15) $$\overline{\lim_{i \to \infty}} \frac{s(i)}{t(i)} = h, \quad 1 \leq h < \infty .$$

Let F(A) be an analytic function of A, and let F(A) when expressed as a polynomial of least degree in A be of degree s in A. Then

(1.16) $$\overline{\lim_{i \to \infty}} \frac{\log \max_j |f_{ij}|}{i^u} \leq \left\{ 1 + (s - 1)h^u \right\} a;$$

the sign of equality in (1.16) may take place.

<u>Theorem 1.2</u> : Let $A = \left[a_{ij} \right]$ be an algebraic semi block matrix satisfying

(1.17) $$\overline{\lim_{i \to \infty}} \frac{\log \max_j |a_{ij}|}{\log i} = a, \quad 0 < a < \infty$$

(1.18) $$\overline{\lim_{i \to \infty}} \frac{\log s(i)}{\log t(i)} = h, \quad 1 \leq h < \infty$$

(1.19) the number of non-zero elements in each row of A is bounded.

Let F(A) be an analytic function of A, and let F(A) when expressed as a polynomial of least degree in A be of degree s in A. Then

$$(1.20) \qquad \overline{\lim_{i\to\infty}} \frac{\log \max_j |f_{ij}|}{\log i} \leq \left\{ 1 + (s-1)h \right\} a;$$

the sign of equality in (1.20) may take place.

Theorem 1.3: Let $A = \left[a_{ij} \right]$ be an algebraic semi block matrix satisfying (1.14) and

$$(1.21) \qquad \overline{\lim_{i\to\infty}} \frac{s(i)}{\left\{ t(i) \right\}^{p}} = h \quad , \quad 1 \leq p < \infty , \quad 0 < h < \infty .$$

Let $F(A)$ be an analytic function of A, and let $F(A)$ when expressed as a polynomial of least degree in A be of degree s in A. Then

$$(1.22) \qquad \overline{\lim_{i\to\infty}} \frac{\log \max_j |f_{ij}|}{i^{pu}} \leq (s-1)h^{u} a;$$

the sign of equality in (1.22) may take place.

When an algebraic matrix A has the minimum equation in (1.2), if we write

$$(1.23) \qquad \frac{1}{f(z)} = \sum_{i=1}^{t} \left\{ \sum_{j=0}^{m_i - 1} \frac{p_{ij}}{(z-u_i)^{j+1}} \right\}$$

then the polynomial of least degree in A representing $F(A)$ is precisely of degree $m-1$ in A if $F(z)$ satisfies the condition [10],

$$(1.24) \qquad \sum_{i,j} \frac{p_{ij}}{j!} F^{(j)}(u_i) \neq 0 .$$

It follows that when A is an algebraic semi block matrix having the minimum equation in (1.2) and $F(z)$ satisfies (1.24), s in the statements of theorems 1.1, 1.2 and 1.3 should be replaced by m-1.

When an algebraic infinite matrix A has the minimum equation in (1.2), a function F(A) satisfying the conditions

(1.25) $F(u_i) \neq F(u_j)$ whenever $i \neq j$,

(1.26) $F'(u_i) \neq 0$ whenever $m_i > 1$

is of particular interest. Such a function is an algebraic matrix of the same degree as A [10] and the matrix A can be expressed as a polynomial in F(A). Considering such functions we have the following results [9] which are derivable from theorems 1.1 and 1.2 respectively.

<u>Corollary 1.1</u> Let $A = \left[a_{ij}\right]$ be an algebraic semi block matrix satisfying (1.14) and (1.15). Let F(A) be an analytic function of A satisfying (1.25) and (1.26). Let s be the degree of the polynomial of least degree in A respesenting F(A) and t be the degree of the polynomial of least degree in F = F(A) for A. Then

$$(1.27) \quad \frac{a}{\left\{1+(t-1)h^{u}\right\}} \leq \overline{\lim_{i \to \infty}} \frac{\log \max_{j} \left|f_{ij}\right|}{i^{u}} \leq \left\{1+(s-1)h^{u}\right\} a ;$$

both of the equality signs in (1.27) may take place.

<u>Corollary 1.2</u>: Let $A = \left[a_{ij}\right]$ be an algebraic semi block matrix satisfying (1.17), (1.18) and (1.19). Let F(A) be an analytic function of A satisfying (1.25) and (1.26). Let s be the degree of the polynomial of least degree in A representing F(A) and t be the degree of the polynomial of

least degree in $P = P(A\ )$ for A. Then

$$(1.28)\qquad \frac{a}{\left\{1+(t-1)\,h\right\}} \leq \overline{\lim_{i\to\infty}}\ \frac{\log \max_j |p_{ij}|}{\log i} \leq \left\{1+(s-1)h\right\}\, a;$$

both of the equality signs in (1.28) may take place.

The deduction of the lower bound in (1.28) from theorem 1.2 makes use of the fact that when the number of non-zero elements in each row of an infinite matrix A is bounded so also is the number of non-zero elements in each row of $P(A)$ where $P(z)$ is a polynomial in z [7].

The condition in (1.14) implies that given a' arbitrarily greater than a, we have

$$(1.29)\qquad |a_{ij}| < K\ e^{a'\,i^u}\ ,\quad i \geq 1\ ,\quad o \leq j \leq s(i)$$

K being some constant ≥ 1. We get the same result in (1.16) if we consider the condition

$$(1.30)\qquad |a_{ij}| \leq K\ e^{a\ i^u}\ ,\quad i \geq 1,\ o \leq j \leq s\ (i);$$

but obviously condition (1.14) is wider (less restrictive) than condition (1.30). In (1.30) the order of magnitude of the elements a_{ij} is related to the row - index i. We may instead, relate the order of magnitude of the elements a_{ij} with the column -index j. Thus in place of condition (1.30) we may impose the condition

$$(1.31)\qquad |a_{ij}| \leq K\ e^{a j^u}\ ,\quad i,\ j \geq 1.$$

It seems, on a first glance, that the result corresponding to (1.31), analogous to that in (1.16), does not require a detailed proof as for the result in (1.16); it does seem that such a result can be easily derived from that in theorem 1.1. In the following we show that this conclusion is quite false.

Condition (1.15) implies that given an h' arbitrary greater than h, there is an i_0 such that

$$(1.32) \qquad s(i) < h' t(i), \quad \text{for all } i \geq i_0 .$$

Since $t(i)$ is always less than i, we then have

$$(1.33) \qquad s(i) < h' i , \quad \text{for all } i \geq i_0 .$$

Recalling that for every i, $a_{ij} = o$ for all $j > s(i)$, we have from (1.31),

$$(1.34) \qquad |a_{ij}| \leq K e^{a \{s(i)\}^u} , \quad i, j \geq 1$$

and so in virtue of (1.33)

$$(1.35) \qquad |a_{ij}| \leq K e^{ah'^u . i^u} , \quad i \geq i_0 .$$

It follows that

$$(1.36) \qquad \overline{\lim_{i \to \infty}} \frac{\log \max_j |a_{ij}|}{i^u} \leq a h'^u .$$

Since h' is arbitrarily chosen greater than h, then

$$(1.37) \qquad \overline{\lim_{i \to \infty}} \frac{\log \max_j a_{ij}}{i^u} \leq a h^u .$$

Comparison of (1.37) with (1.14) apparently shows that the result corresponding to (1.31) can be derived from that in (1.16) by writing ah^u in place of a. Doing so we get the result that when in the statement of theorem 1.1, condition (1.14) is replaced by condition (1.31), (1.16) is replaced by

$$(1.38) \qquad \overline{\lim_{i \to \infty}} \frac{\log \max_j |f_{ij}|}{i^u} \leq \{1+(m-1)h^u\} ah^u$$