ON THE USE OF LATIN SQUARES FOR DESIGNING CARRY-OVER EXPERIMENTS

A THESIS

Submitted in partial fulfilment of the requirements for the award of the M. Sc. Degree

By

NADER LABIB BASILY

B.Sc. Hons. 1972

Supervised by

Professor Abd El-Azim Anis
Professor of Mathematical Statistics
Department of Pure Mathematics
Faculty of Science
Ain Shams University

Professor Salib Rafael Salib
Professor of Mathematical Statistics
Department of Statistics
Faculty of Economics and Political Sciences
Cairo University

Ain Shams

JULY **1975**

No man ought to be discouraged if the experiments he puts in practice answer not his expectations, for what succeeds pleases more, but what succeeds not many times, informs no less. "BACON"

بسم الله الرحمن الرحيم

ACKNOWLEDGEMENT

I wish to thank Professor ABD EL-AZIM ANIS, Professor of Mathematical Statistics, Faculty of Science, Ain Shams University for his fruitful discussion, and his encouragement during the preparation of this thesis.

I wish to express my sincere gratitude to Professor SALIB RAFAIL SALIB, Professor of Mathematical Statistics, Faculty of Economics and Political Sciences for suggesting the problem and his continuous guidance and help.

M.Sc. COURSES

STUDIED BY THE AUTHOR (1972-73)

1- Theory of Probability	2 hrs. weekly
2- Theory of Storage	2 hrs. weekly
3- Stochastic Processes	2 hrs. weekly
4- Design of Experiments	2 hrs. weekly
5- Statistical Inference	2 hrs. weekly

CONTENTS

---o0o---

PREFACE

The thesis aims at utilizing latin squares in the design and analysis of carry-over effects of treatments in block experiments. It also aims at preparing computer programs for the processing of data in some of the designs presented in the thesis.

The thesis is composed of three parts, the first of which introduces general mathematical approach for block experiments with estimators of parameters in the linear model adopted for these experiments and the types of statistical tests of linear hypotheses under investigation. This first part includes computer programs for computation of estimates and conducting statistical tests. It also throws light on some aspects of spoilt experiments.

Moreover, this first part is composed of three chapters, the first of which is concerned with a general mathematical approach to the analysis of block experiments, the second deals with properties and methods of designing block experiments, and the last chapter deals with spoilt experiments.

In the second part emphasis is put on two-way, three-way and multiple-way classified experiments. We introduce basis of designs and methods of analysis.

New contribution in hyper cubic latin designs is also introduced. The second part is composed of three chapters, the first of which considers two-way classified experiments, the second three-way, and the third multiple classifactory designs. The author introduces some basis of designs of his experiments and their methods of analysis and tests of hypotheses involved. Computer programs are also supplied in this part.

In the third part emphasis is being put on carry over designs using latin squares. Methods of estimations of carry over effects are introduced as well as the statistical tests of hypotheses involving carry over effects.

PART ONE

THE DESIGN AND ANALYSIS OF BLOCK EXPERIMENTS

1.1 Introduction

A block experiment is an experiment whose purpose is to estimate the effects of certain treatments applied to some material which is grouped into relatively homogeneous blocks. In the analysis of the results of such experiments care must be taken to ensure that decrease in variability of response arising from the grouping is reflected in the treatment estimates. Experiments of this kind constitute the majority of experiments which are subject to statistical analysis.

The purpose of this part is three-fold. First, it is intended to derive the common forms of analysis of such experiments using matrix methods. The systematic use of this condensed notation enables the common features of as well as the differences between the various analysis used to be better appreciated. It also brings out the essential similarity of the analysis for different experimental arrangements.

The second purpose is to prepare the computations necessary in these analysis in a form suitable for use on automatic calculating machines. Such machines, once given their instructions, will be able to analyse any experimental data considerably more quickly than human computers, and with the added advantage of relieving the tedium of routine calculation. However, before this is a practical possibility, it is necessary to prepare one set of instructions which will enable the machine to analyse all possible experiments. Unless this is done the labour of devising afresh a set of instructions for each individual design would destroy the advantage of the use of such machines over manual calculation.

The matrix presentation of the accepted methods of analysis leads immediately to the pattern of calculation which can form the basis of such a set of instructions.

Obviously, the use of the calculating machines may serve the purpose but is far inferior to the use of an electronic computer.

i.e., if the size of computational work is great, we have proposed a Fortran programs to reduce it and make it practically applicable to be encountered in practical life.

The last function of this part is to present the problem of design. It is widely recognized that the purpose of any experiment of this kind is to estimate the effects of the treatments as accurately as possible.

It will also be shown how the problems of the analysis of spoilt experiments can be tackled using the same basic pattern of calculation.

Matrix Notations Adopted

This section is devoted to a brief summary of the properties of the principal mathematical tools we shall use in the following chapters. Readers who are familiar with matrices are advised not to omit this section altogether, as although the results are all reached by elementary means, certain non-standard notations are introduced for convenience.

A vector whose elements are all units is denoted by $\underline{1}$. If the length (i.e. number of rows) of the vector requires indication, this is placed as a subscript in square bracket, e.g. $\underline{1}_{[n]}$, a square matrix with all off-diagonal elements zero is called diagonal. If the diagonal elements are arranged as a vector $\underline{x}$, then the diagonal matrix is denoted by X^{δ} and the inverse of this diagonal matrix is denoted by $X^{-\delta}$.

The main problem arising in the following chapters is the inversion of certain square symmetric matrix, we have

$$(1^{\delta}+ ab)^{-1} = 1^{\delta}- a(1^{\delta}+ ba)^{-1} b$$

this can be proved by direct multiplication.

The extreme case when $\{a\}$ is a column vector (and hence b is a row vector) is very useful. In this case $1^{\delta}+ ba$ reduces to a scalar and the inversion can be computed as an example. Suppose $a = \lambda \underline{1}$, $b = \underline{1}'$ then if the vectors are of length n then,

$$(1^{\delta} + \lambda \underline{1}\ \underline{1}')^{-1} = 1^{\delta} - \frac{\lambda}{1 + \lambda n}\ \underline{1}\ \underline{1}'$$

Matrices of this type constantly occur and we define the left-hand side of this equation as

$$\{ z_n (\lambda) \}^{-1}, \text{ then}$$

$$\{ z_n (\lambda) \}^{-1} = z_n \left(\frac{-\lambda}{1 + \lambda n} \right)$$

Also, we must note that $| z_n (\lambda) | = 1 + n \lambda$

CHAPTER ONE

1.2- ANALYSIS OF BLOCK EXPERIMENTS

1.2.1- The general linear set-up experiments and its analysis:

Suppose that an experiment gives rise to n observations, we require to describe the distributions from which these can be regarded as a sample, we assume the following properties:-

i) the observations are uncorrelated with each other.

ii) they all have equally variance σ^2.

iii) their mean values can be expressed as known linear compounds of a set of p parameters and $p \leqslant n$.

iv) no mean can be expressed as a linear compound of less than p other means.

Suppose we have a vector $\underline{y}$ denotes the observations and that $\underline{\theta}$ is the vector formed from the parameters.

Then the linear set-up is given by the equations

$$E(\underline{y}) = a\,\underline{\theta}, \qquad V(\underline{y}) = \sigma^2\, 1^{\delta}$$

where $\{a\}$ is n x p matrix of rank p, of known coefficients in the various linear compounds denoted by the design matrix, and σ^2 is the variance of any individual observation.
The fundamental result for the whole theory of design of experiments as developed here is the Gauss-Markoff theorem, this states that :-

" the unbiased linear estimate of minimum variance of any parameter is that given by the method of least squares".

This result is so well known that its proof will not be produced here, the explicit expression for the best estimate of $\underline{\theta}$ is denoted by $\hat{\underline{\theta}}$, and

$$\hat{\underline{\theta}} = (a'a)^{-1} a'\underline{y}$$

Suppose we write

$\underline{y} = a\underline{\theta} + \underline{\epsilon}$, taking the expectation for both sides then $E(\underline{\epsilon}) = o$

$$\hat{\underline{\theta}} = (a'a)^{-1}a'\underline{y} = (a'a)^{-1}a'(a\underline{\theta} + \underline{\epsilon})$$
$$= (a'a)^{-1}(a'a)\,\underline{\theta} + (a'a)^{-1}\,a'\underline{\epsilon}$$
$$= \underline{\theta} + (a'a)^{-1}\,a'\underline{\epsilon}$$

then

$E(\hat{\underline{\theta}}) = \underline{\theta}$, means $\hat{\underline{\theta}}$ is an unbiased estimate for $\underline{\theta}$.

also, $\hat{\underline{\theta}} - \underline{\theta} = (a'a)^{-1}\,a'\underline{\epsilon}$, $\underline{y} - a\,\underline{\theta} = \underline{\epsilon}$

$$v(\underline{y}) = E\left\{(\underline{y}-a\,\underline{\theta})(\underline{y} - a\,\underline{\theta})'\right\} = E(\underline{\epsilon}\,\underline{\epsilon}') = I^{\delta}\,\sigma^2 ,$$

$$v(\hat{\underline{\theta}}) = E\left\{(\hat{\underline{\theta}}-\underline{\theta})(\hat{\underline{\theta}}-\underline{\theta})'\right\} = E\left\{(a'a)^{-1}a'\,\underline{\epsilon}\,\underline{\epsilon}'a(a'a)^{-1}\right\}$$
$$= (a'a)^{-1}a'\,E(\underline{\epsilon}\,\underline{\epsilon}')\,a(a'a)^{-1} = (a'a)^{-1}\,\sigma^2$$

Thus the variance matrix of the estimates of parameter vector $\underline{\theta}$ which form the centre of intrest in the design of experiments is immediately calculated from the matrix $\{a\}$ (referred to as the design matrix), and the variance of an individual observation. This latter is often beyond the control of the experimenter and the problem of design must involve the choice of $\{a\}$. Before the suitability of any chosen experiment can be finally settled the value of $(a'a)^{-1}$ must be known, and its calculation is the burden of the work of the analysis. Thus the value of the variance matrix determined during the preliminary design stage of the experiment should be recorded for use in the analysis. In any actual experiment σ^2 must be estimated so that the order of the errors in the parameter estimates can be determined. The usual estimate of σ^2 is determined from the minimized sum of squares:

$$S = (\underline{y} - a\hat{\underline{\theta}})' (\underline{y} - a\hat{\underline{\theta}})$$

after a little routine algebra

$$= \underline{y}'(1^{\delta} - a(a'a)^{-1}a')\underline{y}$$

$$= \underline{y}'M\,\underline{y} \quad \text{say}$$

where $M = 1^{\delta} - a(a'a)^{-1} a'$ is symmetric and idempotent matr

$$E(S) = \text{trace}\,(M.D(\underline{y})) + \left[(E(\underline{y}))'\, M\, (E(\underline{y}))\right]$$

but $\quad (E(\underline{y}))'.M.(E(\underline{y})) = 0$

hence

$$E(S) = \text{trace}\,(M.D(\underline{y}))$$