ON HURST PHENOMENA IN STORAGE

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INTRODUCTION

In hydrological research, one often meets these types of ranges of partial sums : the crude range R_n, the adjusted range R_n^{*} , and the rescaled adjusted range R_n^{**}; the last one being the most important of them in hydrological recent research.

In the following, these ranges will be investigated in more detail :
The range of partial sums of random variables is relevant to the design capacity of a water reservoir, and to the optimum rules of release once a reservoir is already built. Hence follows the importance attached by research civil engineers to the statistical properties of the range of partial sums of random variables.

In order to explain this, let us denote the consecutive annual into a reservoir over a period of n years by

$$X_1, X_2, \ldots, X_n$$

where

X_1 , X_2, ..., X_n are random variables having the same statistical properties.

Let the initial contents of the reservoir be a, and the consecutive annual abstractions over the same period be :

$$Z_1, Z_2, \dots, Z_n$$

The reservoir and its initial contents being supposed sufficiently large, the successive net annual contents at the end of each water year will be

$$a,\ a + Y_1 - Z_1,\ a + Y_1 + Y_2 - Z_1 - Z_2,\ \dots,$$
$$,\ a + Y_1 + \dots + Y_n - Z_1 - \dots - Z_n$$

If we denote $x_i = \sum_{r=1}^{i} (Y_r - Z_r)$, then the last expression becomes :

$$a,\ a + x_1,\ a + x_2,\ \dots,\ a + x_n \qquad (1)$$

We denote also the largest of these quantities by $a + M_n$ and the smallest by $a + L_n$. If we neglect the fluctuations of levels within each year and consider only the levels at the end of each year namely the quantities (1), the critical conditions on the reservoir to avoid are :

a) Spilling and

b) Completely emptying.

The factors which control these critical conditions in the present case are :

a) reservoir capacity = $a + L_n$

b) $a + L_n > 0$

That is, the required reservoir capacity is

$$M_n - L_n = \max (0, x_1, x_2, \ldots, x_n) - \min (0, x_1, x_2, \ldots, x_n) \qquad (2)$$

The quantity $(M_n - L_n)$ defined in (2) is the range of the accumulated sums of the variables

$$0, Y_1 - Z_1, Y_2 - Z_2, \ldots, Y_n - Z_n.$$

In hydrological contexts the explicit reference to accumulated sums is usually omitted, and one speaks simply of range.

In studies of this type the simplification is often made by taking the abstracted quantities $Z_1, Z_2, \ldots, Z_n$ to have a constant value **Z** independent of n

$$M_n - L_n = \max (0, Y_1 - Z, Y_1 + Y_2 - 2Z, \ldots, Y_1 + \ldots + Y_n - nZ) -$$

$$- \min (0, Y_1 - Z, Y_1 + Y_2 - 2Z, \ldots, Y_1 + \ldots + Y_n - nZ)$$

this being the range of accumulated sums of the quantities:

$$0,\ Y_1 - Z,\ Y_2 - Z,\ \ldots,\ Y_n - Z.$$

In this case, the range is often called, in hydrological references, the "crude range" and is denoted by R_n.

It is also of interest in hydrology to consider the special case when the final contents of the reservoir, at the end of the n-year periods, exactly equal the initial content.

In this case

$$Y_1 + Y_2 + \ldots + Y_n - nZ = 0$$

so that

$$Z = \overline{Y}_n$$

In this case the range $M_n - L_n$ is called the "adjusted range" and is denoted by R_n^*

$$R_n^* = \max(0, Y_1 - \overline{Y}_n,\ Y_1 + Y_2 - 2\overline{Y}_n, \ldots, Y_1 + Y_2 + \ldots + Y_{n-1} - (n-1)\overline{Y}_n) -$$
$$- \min(0, Y_1 - \overline{Y}_n,\ Y_1 + Y_2 - 2\overline{Y}_n, \ldots, Y_1 + Y_2 + \ldots + Y_{n-1} - (n-1)\overline{Y}_n) \quad (3)$$

The magnitude of the adjusted range of an n-year record as defined in (3) is clearly related to the inherent variability of the data. Highly variable data will usually posses a large adjusted range, while relatively invariable data will posses only a small adjusted range. In order to allow for comparisons between runs of data from a given river, or between sets of data from different rivers.

Hurst (1951) introduced the idea of scaling the adjusted range $M_n - L_n$ in (3) by dividing by the sample standard deviation D_n of the n inflows $Y_1, Y_2, \ldots, Y_n$. The resulting relation :

$$R_n^{**} = R_n^{*} / D_n \qquad (4)$$

where

$$nD_n^2 = \sum_{i=1}^{n} (Y_i - \bar{Y}_n)^2 \qquad (5)$$

is called the "rescaled adjusted range" or the Hurst range. It is clear that the study of R_n^{**} is more difficult and less tractable from a mathematical point of view, than the ones of R_n or R_n^{*}.

The object of this thesis is to throw light on the Hurst effect in storage and to investigate the different models which deal with it.

The thesis is divided into four chapters. In the first we introduce Hurst's empirical law, its modified, and theoretical work on crude, adjusted and rescaled ranges.

The second chapter relies mainly on the results obtained by some authors (Feller(1951), Anis and Lloyd(1953), Solari and Anis (1957), Boes and Salas - La Cruz (1973), and Anis and Lloyd (1974)).

In chapter three, we proceed a further step to discuss the expected crude range and the expected adjusted range of partial sums of gamma (m) summands (Anis and Lloyd, 1974, (3)). Also, in this chapter, we apply Spitzer (22) lemma to find, for the first time, the expected crude and adjusted ranges for non-central χ^2 summands. The case of central χ^2 summands can be obtained as a special case from our general treatment by letting the parameter of the non-central χ^2 tends to zero. The results obtained by using this procedure are exactly similar to the results obtained before in reference (3).

In the last chapter an investigation for all the results obtained before by different authors is given and it was found that the results obtained by using the present procedure is quite good on comparing them with those obtained before in reference (3).

However one must admit that all the results obtained so far including the present results does not agree completely with Hurst effect. We suggest here that a remedy for this might be that a new storage distribution must be introduced for the beginning of the treatment. We hope to do this in the nearest future.

CHAPTER I

INTRODUCTION TO HURST EFFECT

This chapter reviews the following subjects :

i- Hurst's law.

ii- The qualitative nature of the conformity of some additive processes to Hurst's law. Degree of approximation to a Hurst-like law.

iii- Theoretical work on crude, adjusted and rescaled ranges.

I-i) Hurst's Law:

The announcement by Hurst[1] of his empirical findings on the time - dependence of the statistic which we now call the Hurst range initiated a large volume of research by mathematicians, probabilists and stochastic hydrologists to examine whether a number of more or less plausible mathematical models of the structure of relevant geophysical time series were consistent with Hurst's law.

Properties of time series that have been emphasised in various types of models of this kind include their correlation and frequency structure, the finiteness or otherwise of their "memory", and their stationarity or otherwise : for a recent survey (see Klemes[2]).

Surprisingly enough, it is still necessary at this stage of study of the Hurst phenomenon, to define with some care what the phenomenon is.

Hurst started by considering the annual flows X_t ($t = 1,2,\ldots$) in various rivers. The sequence $\{X_t\}$ of flows on a given stream, if fed into a sufficiently large reservoir from which no abstraction took place, would change the initial level of water stored by an amount:

$$S_r = X_1 + X_2 + \ldots + X_r$$

at the end of the r- th year ($r = 1,2,\ldots$), leading to a net increase $S_n = n\bar{X}_n$ (which might in fact be negative) after n years. (Here we neglect seepage, evaporation and other losses. We take $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$).

The imposition for each of n years of a constant annual draft of amount $\bar{X}_n$ would therefore restore this idealized reservoir to its initial level, this n - year regulation resulting in the reservoir having contents

$$_nS_r^* = S_r - r\,\bar{X}_n$$

in year r ($r = 1,2,\ldots, n$), measured from the initial level as origin, where a negative value of $_nS_r^*$ means a deficit, and where $_nS_n^* = 0$.

If the inflows X_t were fed into a finite reservoir of capacity $U_n - L_n$ where U_n is the greatest, and L_n the smallest, of the quantities

$$ {}_nS_1^{*}, \ {}_nS_2^{*}, \ \ldots, \ {}_nS_{n-1}^{*}, \ 0 $$

the reservoir would just suffice, in the sense that it would at some time (s) be completely full and at some times (s) be completely empty, without ever overflowing or failing to supply the required draft. The so-called n-year adjusted range $U_n - L_n$ is a random variable of considerable engineering interest, since knowledge of its statistical behaviour would be relevant to the design of an annual reservoir fed by the river in question.

Hurst recognised the importance of this variable and saw that its dependence on the scale of the fluctuations in the annual increment X_r could be eliminated by dividing by a suitable scaling factor D_n.

He took D_n to be the sample standard deviation of the X_r :

$$ D_n = \sqrt{\frac{\sum_{1}^{n} (X_r - X_n)^2}{n}} \qquad \text{(I-1.1)} $$