STABILITY
OF MOTION IN CRITICAL CASES

THESIS

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OF

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By

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TO MY HUSBAND

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بسم الله الرحمن الرحيم

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CONTENTS

Summary

One of the recent studies in the theory of differential equations is that of the stability of motion for a system of differential equations of the form:

$$\frac{dx}{dt} = f(t,x) \qquad (1)$$

where t is real (usually denoting time) and x is a vector with components $x_1, x_2, \dots, x_n$. The vector function $f(t,x)$ is assumed to be continuous with respect to the variable t and the components of x, and has continuous partial derivatives with respect to the components of x. Moreover it satisfies the condition

$$f(t,o) = o \qquad (2)$$

Thus the system of differential equations (1) satisfies the zero solution

$$x \equiv o \qquad (3)$$

In the first chapter of this thesis, we give the definitions and theorems of stability and instability of motion of the zero solution of the system of differential equations (1).

The system is written in the first approximation form:

$$\frac{dx}{dt} = Ax + X(t,x) \qquad (4)$$

Where A is a square matrix of order n, and the vector function X(t,x) is continuous with continuous partial derivatives with respect to the components x, and satisfying the two conditions.

$$X(t,o) = o \ , \ \frac{\partial X(t,o)}{\partial x} = o \qquad (5)$$

We give the results which have been obtained as regards the stability of solutions of the non-linear differential equations (4). Denoting by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the roots of the characteristic equation.

$$\det (A - \lambda E) = o \qquad (6) ,$$

and imposing rather weak conditions on X(t,x), it has been shown that when

$$Re(\lambda_i) < o \qquad (i = 1,\ldots,n) \qquad (7) ,$$

the zero solutions of the linear system

$$\frac{dX}{dt} = AX \qquad (8)$$

and also the zero solution of the non linear system of equations (4) are asymptotically stable. When, on the other hand

$$Re(\lambda_i) > 0$$

for at least one value of i (i.e. when the real part of one of the roots of the characteristic equation (6) is positive) then the zero solution (3) is unstable for both the linear system (8) and the non linear system (4). This leads us to consider the critical case when one of the roots of the characteristic equation (6) is equal to zero; all other roots having negative real parts. For example when one root of (6) is zero, Lyapunov studied the stability of the zero solution of the non linear system of equations (4) in each of the following two cases:

1. When the function X(t,x) is periodic in t.
2. When X is independent of the time t i.e. when it is of the form X(x).

In both these cases Lyapunov assumed that the function is in the form of an infinite power series of x, in which the terms lower than the second are absent.

Krasowosky, considered the stability of the zero-solution of the system

$$\left.\begin{aligned} \frac{dy}{dt} &= Y(y,x) \\ \frac{dx}{dt} &= Ax + X(y,x) \end{aligned}\right\} \qquad (9)$$

where y is real and x a vector with components $x_1, x_2, \ldots x_n$; A being a matrix of order n, the characteristic equation of which has roots with negative real parts.

Krasowsky assumed that X,Y are continuous function with respect to their orguments and have continuous partial derivatives satisfying the following conditions

$$\left.\begin{aligned} Y(o,o) &= o, \quad \frac{\partial Y(o,o)}{\partial y} = o, \quad \frac{\partial Y(o,o)}{\partial x} = o \\ X(o,o) &= o, \quad \frac{\partial X(o,o)}{\partial y} = o, \quad \frac{\partial X(o,o)}{\partial x} = o \end{aligned}\right\} \qquad (10)$$

He proved that, in this case, there exists a function $\phi(y)$, depending essentially on Y. Which plays a principal role in the study of the stability of the zero solution of the system (9). He studied the stability of motion on the assumption that $\phi(y) \neq o$ for $y \neq o$ and obtained the following results

(i) If $y\phi(y) < 0$, $y \neq 0$, for $|y| < h$, then the zero solution of the system (9) is asymptotically stable.

(ii) If $\phi(y) > 0$ for $y > 0$,

or $\phi(y) < 0$ for $y < 0$,

then the zero solution is unstable. In 1966 Dr. Mur Nassif studied the stability of motion for each of t two system of differential equations.

$$\left.\begin{aligned} \frac{dy}{dt} &= Y(y,x) \\ \frac{dx}{dt} &= Ax + X(y,x) \end{aligned}\right\} \qquad (9)$$

$$\left.\begin{aligned} \frac{dy}{dt} &= Y(t,y,x) \\ \frac{dx}{dt} &= Ax + X(t,y,x) \end{aligned}\right\} \qquad (11)$$

under conditions similar to those of Krasowosky as regards the two functions X,Y.

For the system (11), X(t,y,x), Y(t,y,x) are assumed to be periodic in t, i.e.

$$\left.\begin{aligned} Y(t+w,y,x) &= Y(t,y,x) \\ X(t+w,y,x) &= X(t,y,x) \end{aligned}\right\} \qquad (12)$$

In this case, as in the previous one, the function $\phi(y)$ is obtained, which, as stated before, plays a principal role in the study of stability of motion.

Dr. Munir Nassif considered the question of stability on the assumption that

$$\phi(y) = 0 \quad \text{for} \quad y \in C ,$$

where C is a set of real numbers having a limiting point at $y = 0$.

Under these conditions on $\phi(y)$, besides the zero solution, there exist for the system of equati(ons) (9) non-zero solutions in the form:

$$y = c \quad , \quad x = u(c)^{+} \qquad (13) ,$$

the system (11) has periodic solutions of the form

$$y = y(t,c) \; , \quad x = (t,c) \qquad (14)^{+}$$

In Chapter II, of this thesis, we treat the question of the stability of motion for the non-zero solutions (13). The following theorem has been proved:

<u>Theorem</u>:

(1) If there exists a sequence of positive numbers

$$y_i \xrightarrow[i \to \infty]{} 0 \; , \text{ such that}$$

+ The function u(c) is defined in chapter II page, 27, the functions y(t,c), x(t,c) are defined in chapter III page 53.

$\phi(y_i) = o \qquad i = 1,2, \ldots.$

and another sequence of negative numbers $\bar{y}_i \xrightarrow[i \to \infty]{} o$

such that $\phi(\bar{y}_i) = o$ for $i = 1,2, \ldots$

then the non zero solution

$$y = o \qquad x = u(c)$$

is stable.

(2) If there is a sequence of positive (negative) numbers $y_i \xrightarrow[i \to \infty]{} o$, such that

$$\phi(y_i) = o \qquad i = 1,2, \ldots.$$

and $\phi(y) > o$ for $y < o$ ($\phi(y) < o$ for $y > o$)

then the solution (13) is stable.

Chapter III deals with the stability of periodic solution. (14). The following theorem is obtained.

<u>Theorem:</u>

(i) If there is a sequence of positive numbers $Z_i \longrightarrow o$ such that

$$\phi(Z_i) = o \qquad i = 1,2, \ldots, m \ldots,$$

and there is a sequence of negative number $\bar{Z}_i \to o$ such that

$$\phi(\bar{Z}_i) = o \qquad i = 1, 2, \ldots \; ,$$

then the solution

$$y = y(t, o) \qquad X = X(t, c)$$

is stable.

(2) If there exist a sequence of positive (negative) numbers $Z_i \longrightarrow o$ such that

$$\phi(Z_i) = o \qquad i = 1, \ldots \; n, \; \ldots ,$$

$$\phi(Z) > o \quad \text{for} \quad Z < o.$$

$$(\phi(Z) < o \quad \text{for} \quad Z > o),$$ then the solution (14) is stable.