ON ROW-FINITE MATRICES
AND ON BASIC SETS OF POLYNOMIALS

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PREFACE

The thesis consists of three chapters. In chapter I we investigate four main convergence properties of basic sets of polynomials associated with non-singular matrix functions F(P,Q) where P and Q are two commutative algebraic row-finite matrices. These convergence properties are effectiveness in an open circle, effectiveness in closed circles, effectiveness at the origin for every function and effectiveness in the whole plane for every entire function. Five theorems are established.

In chapter II we investigate three main convergence properties of basic sets of polynomials of two complex variables associated with non-singular matrix functions F(P,Q). These convergence properties are effectiveness in an open disc, effectiveness in closed discs, and effectiveness at the origin. Three theorems are established.

In chapter III we deal with two problems. The first is the order of magnitude of the elements of a matrix function F(A,B) where A and B are two commutative algebraic row-finite matrices whether general row-finite ones or semi-block matrices. Eleven theorems are given in this concern. The second problem is the order, and type on a circle, of basic sets of polynomials associated with non-singular matrix functions F(P,Q), whether P and Q are general row-finite matrices or semi-block matrices. Eight theorems are given in this concern.

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CONTENTS

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CHAPTER I

ON THE EFFECTIVENESS OF BASIC SETS OF POLYNOMIALS ASSOCIATED WITH NON-SINGULAR MATRIX FUNCTIONS OF TWO COMMUTATIVE ALGEBRAIC ROW-FINITE MATRICES

The definition of an analytic function $F(A,B)$ of two commutative algebraic matrices A and B, belonging to an associative field, is explained in §1. A few words on basic sets of polynomials are given in § 2. In §§ 3–7 we investigate the convergence properties of a basic set associated with a non-singular analytic matrix function $F(P,Q)$ where P and Q are commutative algebraic row-finite matrices satisfying some certain conditions. § 3 is concerned with effectiveness in an open circle $|z| < R$. § 4 is concerned with effectiveness in closed circles $|z| \leqslant R$ where R belongs to an open interval $a \leqslant R < b$. § 5 deals with effectiveness at the origin for every function. § 6 deals with effectiveness, in the whole plane, for every entire function. Lastly § 7 deals with effectiveness in closed circles $|z| \leqslant R$ where R belongs to a closed interval $a \leqslant R \leqslant b$.

1. Functions of algebraic infinite matrices

An infinite matrix P is said to be self-associative when the power matrix P^r exists for every positive integer r, and has a unique matrix value independent of the order of multiplication of the factors. An infinite matrix P is said to be algebraic if it is self-associative and satisfies an equation (identity) of the form

(1.1) $$k_0 I + k_1 A + \dots + k_m A^m = 0$$

where the coefficients k_r, $0 \leqslant r \leqslant m$ are complex scalars, and I and O are the infinite unit matrix and the infinite zero matrix respectively. If this equation is the equation of least degree satisfied by the matrix A, A is said to be algebraic of degree m. Such equation, of least degree, is unique save for a multiplying constant and is called the minimum equation of A.

The roots of the scalar equation $f(z) = 0$, where $f(A) = 0$ is the minimum equation of A, are called the scalar roots of A. The set of sclar roots of A is called the spectrum of A.

When F(z) is a function of a complex variable z, analytic in a domain D within which the spectrum of an algebraic infinite matrix A lies, then F(A) may be defined by the so-called Hermite-Sylvester interpolation polynomial for $F(A)$. This definition due to Makar and Sakr [13] which is a generalization of Schwerdtfeger's definition [20, see also 18, § 2.6] of a function of a square matrix runs as follows.

Let the minimum equation of the algebraic matrix A be

$$(1.2) \qquad f(A) = \prod_{i=1}^{t} (A-u_i I)^{m_i} = 0, \quad \sum_{i=1}^{t} m_i = m.$$

The rational function $1/f(z)$ may be written as a sum of fractions of the form

$$(1.3) \qquad \frac{1}{f(z)} = \sum_{i=1}^{t} p_i(z)/(z-u_i)^{m_i} .$$

The polynomials $f_i(z)$, $i = 1,2,\ldots,t$ are now defined by

$$(1.4) \qquad f_i(z) = p_i(z)f(z)/(z - u_i)^{m_i} .$$

Then we have for $F(A)$ the polynomial representation

$$(1.5) \qquad F(A) = \sum_{i=1}^{t} f_i(A) \left\{ \sum_{p=0}^{m_i-1} (A-u_i I)^p F^{(p)}(u_i)/p! \right\} .$$

The polynomials $f_i(A)$, $i = 1,2,\ldots, t$ are called the co-variants of A.

The polynomial representation in (1.5) for $F(A)$ can be reduced by the use of the minimum equation to a polynomial of degree at most m-1 in A. Indeed when the minimum equation of A is written in the form

(1.6) $$f(A) = A^m + k_1 A^{m-1} + k_2 A^{m-2} + \dots + k_m I = 0$$

we have for $F(A)$ the polynomial representation

(1.7) $$F(A) = a_1 A^{m-1} + a_2 A^{m-2} + \dots + a_m I$$

where

(1.8) $$a_r = \frac{1}{2\pi i} \int_C \left(w^{r-1} + k_1 w^{r-2} + \dots + k_{r-1} \right) \frac{F(w)}{f(w)} \, dw \, ,$$

$r = 1,2,\dots,m$; the integrals being taken round a contour C, lying in the domain in which $F(w)$ is analytic, and enclosing the spectrum of A.
This result due to Sakr [19, pp. 65-66] is a generalization to algebraic infinite matrices, of Fantappie's result [2] for square matrices.

Let A be an algebraic infinite matrix having the minimum equation in (1.2). Let B be an algebraic infinite matrix which commutes with A and let B have the minimum equation

(1.9) $$g(B) \equiv \prod_{j=1}^{s} (B - v_j I)^{n_j} = 0 \, , \quad \sum_{j=1}^{s} n_j = n \, .$$

(We always assume that A and B belong to an associative field $\mathfrak{F}$, for example, the associative field of row-finite matrices.) Let $F(z,w)$ be a function of two complex variables z and w analytic in a domain D which is the scalar product of two domains D_1 and D_2. Let the spectrum of A lie in D_1 and that of B lie in D_2. Let $f_i(A)$, $i = 1,2,\ldots,t$ be the covariants of A, as defined above, and let $g_j(B)$, $j = 1,2,\ldots, s$ be the covariants of B, similarly defined. Let

$$F^{(p,q)}(z,w) = \frac{\partial^{p+q}}{\partial z^p \partial w^q} F(z,w). \tag{1.10}$$

Then the matrix function $F(A,B)$ is defined [6] by the interpolating polynomial

$$F(A,B) = \sum_{i=1}^{t} \sum_{j=1}^{s} f_i(A)\, g_j(B) \tag{1.11}$$

$$\times \left\{ \sum_{p=0}^{m_i-1} \sum_{q=0}^{n_j-1} (A-u_iI)^p (B-v_jI)^q \, F^{(p,q)}(u_i,v_j)/p!q! \right\}.$$

Making use of the minimum equations of A and B this interpolation polynomial for $F(A,B)$ can be reduced to a polynomial of degree at most $m-1$ in A and of degree at most $n-1$ in B.

When A and B are two commutative algebraic infinite matrices having the minimum equations in (1.2) and (1.9) respectively, and $F(z,w)$ is analytic in $D \equiv D_1 \times D_2$, with the spectra of A and B lying in D_1 and D_2 respectively, then the mixed partial derivative

$$(1.12) \qquad F^{(\alpha,\beta)}(A,B) \equiv \frac{\partial^{\alpha+\beta}}{\partial A^{\alpha}\partial B^{\beta}} F(A,B)$$

(exists and) is given by [6]

$$(1.13) \qquad F^{(\alpha,\beta)}(A,B) = \sum_{i=1}^{t} \sum_{j=1}^{s} f_i(A)\, g_j(B)$$

$$\times \left\{ \sum_{p=0}^{m_i-1} \sum_{q=0}^{n_j-1} (A-u_iI)^p (B-v_jI)^q\, F^{(p+\alpha,q+\beta)}(u_i,v_j)/p!q! \right\} .$$

This is true for all integral $\alpha,\beta \geqslant 0$. Therefore the function $F(A,B)$ is called an analytic function of the two commutative algebraic matrices A and B.

We note that when B is the (infinite) unit matrix, then B is an algebraic matrix of degree 1 which commutes with A. Also B has one scalar root namely 1, and this scalar root is simple and the associated covariant is the unit matrix. In such a case (1.11) takes the following form with $\emptyset(z,w)$ written in place of $F(z,w)$.

$$(1.14) \quad \emptyset(A,I) = \sum_{i=1}^{t} f_i(A) \left\{ \sum_{p=0}^{m_i-1} (A-u_iI)^p\, \emptyset^{(p,0)}(u_i,1)/p! \right\} .$$

Writing $F(z)$ for $\emptyset(z,1)$, (1.14) becomes (1.5).

2. Basic sets of polynomials

A set of polynomials $\{p_n(z)\} \equiv p_o(z), p_1(z), p_2(z), \ldots$ is said to be a basic set [21] when every polynomial can be uniquely expressed as a finite linear combination of the elements of the set. In particular

(2.1) $$z^n = \sum_i \pi_{ni}\, p_i(z), \qquad n = o,1,2,\ldots .$$

When N_n the number of non-zero coefficients π_{ni} in the representation (2.1) satisfies the condition

(2.2) $$N_n^{\frac{1}{n}} \longrightarrow 1 \qquad \text{as} \quad n \longrightarrow \infty$$

the set is called a Cannon set, otherwise it is a general basic set. The matrix $\Pi = [\pi_{ij}]$ is called the matrix of operators associated with the set and is the unique row-finite reciprocal of the matrix of coefficients $P = [P_{ij}]$ where

(2.3) $$p_i(z) = \sum_j P_{ij}\, z^j \quad , \qquad j = o,1,2,\ldots \quad .$$

Given a function

(2.4) $$F(z) = \sum_{n=o}^{\infty} a_n z^n$$

there is formally an associated basic series

(2.5) $$\sum_{n=o}^{\infty} a_n \left\{ \sum_i \pi_{ni} p_i(z) \right\} = \sum_{n=o}^{\infty} \alpha_n p_n(z) \; .$$

when this associated series converges uniformly to $F(z)$ in some domain it is said to represent $F(z)$ in that domain. The convergence properties of basic sets are classified according to the classes of functions represented by their associated basic series and also according to the domains in which they are represented.

Writing

(2.6) $$A_i(R) = \max_{|z|=R} |p_i(z)|$$

(2.7) $$\omega_n(R) = \sum_i |\pi_{ni}| A_i(R)$$

some main convergence properties of a Cannon basic set depend entirely on the value of the expression

(2.8) $$\lambda(R) = \overline{\lim_{n\to\infty}} \{\omega_n(R)\}^{\frac{1}{n}} .$$

On the other hand, writing

(2.9) $$f_n(R) = \max_{i,j} \left\{ \max_{|z|=R} |\pi_{ni}p_i(z) + \pi_{n,i+1}p_{i+1}(z) + \ldots + \pi_{nj}p_j(z)| \right\}$$

the corresponding convergence properties of a general basic set depend entirely on the value of the expression

(2.10) $$\chi(R) = \overline{\lim_{n\to\infty}} \{f_n(R)\}^{\frac{1}{n}} .$$

In the case of a Cannon set $\chi(R) = \lambda(R)$ for all R. But in general

(2.11) $$R \leqslant \chi(R) \leqslant \lambda(R) .$$

A basic set $\{p_n(z)\}$ whose matrix of coefficients P is an algebraic matrix is called [10] an algebraic basic set.

3. Effectiveness in an open circle $|z| < R$

A basic set of polynomials $\{p_n(z)\}$ is said to be effective in a circle $|z| < R$ when the basic series associated with every function F(z) which is regular in $|z| < R$ represents F(z) in $|z| < R$, [21, p. 11]. The necessary and sufficient condition that a Cannon set is effective in $|z| < R$ is that [21, th. 15, p. 26]

(3.1) $$\lambda(r) < R \quad \text{for all } r < R.$$

For general basic sets the condition is [21, Th. 29, p. 38]

(3.2) $$\chi(r) < R \quad \text{for all } r < R .$$

Writing

(3.3) $$A(R) = \overline{\lim_{n\to\infty}} \left\{ A_n(R) \right\}^{\frac{1}{n}}$$

the condition

(3.4) $$A(r) < R \quad \text{for all } r < R$$

does not imply the effectiveness of the set $\{p_n(z)\}$ in $|z| < R$. For considering the basic set $\{p_n(z)\}$ defined by

(3.5) $$\begin{cases} p_0(z) = 1 \\ p_n(z) = -z^{n-1} + z^n , \quad n \geqslant 1 \end{cases}$$