ON SOME TOPICS OF INFINITE MATRICES

THESIS SUBMITTED IN PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR THE AWARD
OF THE M.SC. DEGREE

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DEPT. OF PURE MATHEMATICS

JANUARY 1974

M.SC. COURSES

STUDIED BY THE AUTHOR (1971-1972)

(i)	Theory of Functions of Matrices	2 hours per week (during the whole academic year)
(ii)	Numerical linear algebra	2 hours per week
(iii)	Abstract algebra	2 hours per week
(iv)	Functional Analysis	2 hours per week
(v)	Partial Differential Equations	2 hours per week

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PREFACE

The thesis consists of three chapters.

The first chapter is mainly a survey to the algebraic part of the subject of algebraic infinite matrices, though the last article is a new addition; in that article we investigate some properties of an algebraic infinite matrix which are retained by the reciprocal matrix.

In chapter II, we establish a number of new results on functions of two commutative algebraic infinite matrices.

Chapter III involves an application of algebraic infinite matrices in "basic sets of polynomials"; a number of new results have been obtained on the order and type on a circle of basic sets associated with functions of an algebraic semi block matrix.

The thesis has been prepared under the kind supervision of Prof. Dr. Ragy H. Makar, to whom I wish to express my sincerest gratitude and thankfulness.

January 1974

CONTENTS

CHAPTER I

ON ALGEBRAIC INFINITE MATRICES

This chapter is mainly a survey to the subject of algebraic infinite matrices, though the last article is a new addition. The given survey considers only problems of algebraic nature. §§ 1-3 deal with the simple algebraic results on these matrices, while §§ 4-7 consider the more deep results. § 8 deals with the definition of functions of an algebraic infinite matrix and with the algebraic properties of such functions. § 9 investigates some properties of an algebraic infinite matrix which are retained by the reciprocal matrix, and some simple results are also deduced.

1. Algebraic infinite matrices

An infinite matrix A is said to be self-associative when the power matrix A^r exists for every positive integer r, and has a unique matrix value independent of the order of multiplication of the factors.

An infinite matrix A is said to be algebraic if it is self-associative and satisfies an equation (identity) of the form:

(1.1) $$k_0 I + k_1 A + k_2 A^2 + \dots + k_m A^m = O$$

where the coefficients k_r, $o \leqslant r \leqslant m$, are complex scalars, and I and O are the infinite unit matrix and the infinite zero matrix respectively. If this equation is the equation of least degree satisfied by the matrix A, A is said to be algebraic of degree* m. Such equation, of least degree, is evidently unique save for a multiplying constant, and is called the minimum equation of A.

This equation of least degree, in some sense, corresponds to the minimum equation of a square matrix, but with the clear understanding that every square matrix has a minimum equation, which is sometimes the characteristic equation itself, whilst amongst infinite matrices only algebraic ones satisfy such algebraic equations.

The roots of the scalar equation $f(z) = o$, where $f(A) = O$ is the minimum equation of A, are called the scalar roots of A. The set of scalar roots of A is called the spectrum of A.

A simple example of an algebraic infinite matrix is the diagonal matrix A in which m distinct numbers $a_1, a_2, \dots, a_m$ are distributed repeatedly and arbitrarily along the

* Originally Raouf H. Makar [15] had used the term "order"

diagonal. It is easily seen that such a matrix is algebraic of degree m and that its minimum equation is

$$(1.2) \qquad (A - a_1 I)(A - a_2 I) \ldots (A - a_m I) = 0 .$$

In particular the scalar matrix whose diagonal elements are all, the number a, has the minimum equation $A - aI = 0$.

Also, a simple example of a non-algebraic matrix is the diagonal matrix A with an infinite number of distinct elements a_i.

Two interesting types of algebraic infinite matrices have been introduced by Ibrahim [6, chapter I].

(i) If the infinite series $a_1+a_2+a_3+ \ldots$ is convergent and its sum is M, then the matrix

$$(1.3) \qquad A = \begin{bmatrix} a_1 & a_1 & a_1 & \cdots \\ a_2 & a_2 & a_2 & \cdots \\ a_3 & a_3 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

is algebraic, and its minimum equation is $A^2 - MA = 0$. Thus associated with every convergent series there is an algebraic matrix of degree 2.

(ii) If f_i, h_i, $i = 1,2,3,\ldots$ are two infinite sequences such that the series $\sum_{i=1}^{\infty} f_i h_i$ is convergent to M, then the matrix

$$(1.4) \qquad A = [a_{ij}] \quad , \qquad a_{ij} = f_i h_j$$

is algebraic, and its minimum equation is $A^2 - MA = 0$.

A further interesting example of an algebraic infinite matrix is the following [10].

(iii) A row (column) permutator is the matrix obtained from the unit matrix by performing a certain permutation on its rows (columns). A permutator corresponding to a permutation factorizable into cycles such that the lengths of these cycles have a finite least common multiple m, is an algebraic matrix of degree m.* For performing such a permutation m times, we ultimately arrive at the identical permutation, and the corresponding matrix is the unit matrix. Thus a permutator corresponding to such a permutation has the minimum equation $A^m = I$. In particular a permutator corresponding to a permutation factorizable to binary cycles only, for example, the permutation interchanging every prime number with its square or every odd number with its subsequent even number satisfies the equation $A^2 = I$.

Some types of matrices related to an algebraic matrix are algebraic matrices. Of these we have [15]:

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* Makar and Sakr consider that there is no common letter between any two cycles, though they have not mentioned this.

matrix A in which the 1st superdiagonal consists of the elements

$$1,\ 0,\ 1,\ 0,\ \ldots$$

and all other elements are zero, has the minimum equation $A^2 = 0$, also the matrix B in which the 1st superdiagonal consists of the elements

$$0,\ 1,\ 0,\ 1,\ \ldots$$

and all other elements are zero, has the minimum equation $B^2 = 0$, yet, as it can be easily verified, the sum matrix $C = A+B$, which is the infinite auxilary unit matrix, is not an algebraic matrix.

In fact A and B may both be algebraic, but the powers of the sum matrix C may not exist. This fact is illustrated by the two matrices

$$(2.1) \qquad A = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

and $B = A'$ (the transpose of A); A and B satisfy respectively $A^2 - A = 0$ and $B^2 - B = 0$ but the square of the sum matrix C does not exist.

There are however cases in [illegible] sum of [illegible] algebraic matrices is certainly an algebraic matrix. Of these [illegible]:

(i) If A is a diagonal matrix involving m distinct numbers $a_1, a_2, \ldots, a_m$ only, and B is another diagonal matrix involving n distinct numbers $b_1, b_2, \ldots, b_n$ only, then A and B are algebraic of degrees m and n respectively. The sum matrix $C = A+B$ is a diagonal matrix involving at most mn distinct numbers

$$a_1+b_1,\ a_1+b_2, \ldots,\ a_1+b_n,\ a_2+b_1,\ \ldots,\ a_m+b_n$$

and so is algebraic of degree at most mn, [15]. But this fact is a particular case of a general theorem to be mentioned later.

(ii) If A and B are algebraic matrices with minimum equations $(A-aI)^m = 0$ and $(B-bI)^n = 0$ respectively, and if A and B commute, then the sum matrix $C = A+B$ is algebraic of degree $m+n-1$ at most. Indeed it is easily seen that

$$\left\{ C - (a + b)I \right\}^{m+n-1} = 0 ;$$

also the fact that the upper bound $m+n-1$ is attainable is illustrated by the following example [10].

Let P and Q be the square matrices

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}.$$

Let A be the infinite matrix consisting of diagonal blocks each of which is the matrix P and zero elements everywhere else, and let B be the infinite matrix consisting of diagonal blocks each of which is the matrix Q and zero elements everywhere else. Then A and B satisfy respectively $(A - I)^2 = 0$ and $(B - I)^2 = 0$. Also it is easily verified that $AB = BA$. The sum matrix $C = A+B$ is an infinite matrix consisting of diagonal blocks each of which is the matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & -1 & 2 \end{bmatrix}$$

and zero elements everywhere else. The minimum equation of C is easily verified to be $(C - 2I)^3 = 0$.

3. The product of algebraic matrices

When A and B are two algebraic matrices, the product matrix $C = AB$ is not necessarily algebraic [15] . In fact the product matrix $C=AB$ may not exist. For if we take the matrix A in (2.1) and take $B = A^T$, then AB does not exist.

If AB exists its powers may not exist. For if we take B to be the matrix A in (2.1) and take $A = B^T$, then the product matrix is

$$C = AB = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

and evidently C^2 does not exist.

Even if $C = AB$ exists and is self-associative, it may not be an algebraic matrix. For considering the matrices A and B at the beginning of § 2, we find that $C = AB$ is the infinite matrix in which the second super-diagonal consists of the elements

$$1, o, 1, o, \ldots$$

and all other elements are zero; such a matrix is easily seen to be non-algebraic.

There are, however, simple cases in which we can assert that the product matrix $C = AB$, of two algebraic matrices A and B, is an algebraic matrix. Of these we have:

(i) The case in which A and B are diagonal matrices. For if the distinct elements of A are $a_1, a_2, \ldots, a_m$ and the distinct elements of B are $b_1, b_2, \ldots, b_n$, then in the product matrix there are at most mn distinct elements

$$a_1b_1,\ a_1b_2,\ \ldots,\ a_1b_n,\ a_2b_1,\ \ldots,\ a_mb_n\ .$$

Thus $C = AB$ is algebraic of degree at most mn. But this fact is a particular case of a general theorem to be mentioned in due time.

(ii) The case in which A and B are two algebraic permutators as defined in § 1. When the corresponding permutations have no letter in common, $C = AB$ is an algebraic permutator whose degree is the least common multiple of the degrees of A and B [10].

4. Polynomials in an algebraic matrix

Polynomials in a self-associative matrix A have the following interesting property [15].

(I) Matrices in P_A, of degree $\geqslant 1$ in A, are either all algebraic or else all non-algebraic.

Indeed when A is an algebraic matrix of degree m, and $P = P(A)$, of degree r in A, is of degree n, then $m/r \leqslant n \leqslant m$. The upper and lower bounds of the degree n of P(A) are both attainable. This fact is illustrated by the following interesting example. If A is the (infinite) diagonal matrix whose (diagonal) elements are the mth roots of unity repeated and distributed arbitrarily, then A satisfies the minimum equation $A^m - I = 0$, while the matrix $P = A^r$, where r is a divisor of m and $m = nr$, satisfies the minimum equation $P^n - I = 0$. Also the matrix $P = A^t$ where t is prime to m, satisfies the minimum equation $P^m - I = 0$.

An immediate corollary of the above property is that the matrix $B = b_oI + b_1A$, where b_o and b_1 are complex scalars and $b_1 \neq o$, is of the same nature as A, and is of the same degree as A, when A is algebraic.

Makar [15] has also proved the following result.

(II) If A and B are two algebraic matrices having the same minimum equation $f(z) = o$ *, and $P(z)$ is a polynomial in z, then the algebraic matrices $P(A)$ and $P(B)$ have the same minimum equation.

A generalization of this result will be mentioned later.

Applying the above result Raouf Makar [15] has investigated the minimum equation of a polynomial $P(A)$ of an algebraic matrix A whose scalar roots are all distinct, and Ragy Makar and Sakr [10] have investigated this same problem but in the more difficult case in which A has multiple scalar roots. Makar's result is that if A is an algebraic matrix of degree m having the distinct scalar roots $a_1, a_2, \ldots, a_m$ and if the polynomial $P(z)$ is such that $P(a_1), P(a_2), \ldots, P(a_m)$ take only r distinct

* i.e. $f(A) = 0$ and $f(B) = o$ are the minimum equations of A and B respectively.