A STUDY OF TUBERIAN THEOREMS

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ABSTRACT

Let us consider a linear transformation

(1) $$t_n = \sum_{k}^{\infty} C_{n,k} S_k$$

of the sequence S_k into the sequence t_n. Then the classic Tauberian theorems state that if a_k(or S_k) satisfies an appropriate Tauberian condition T, then convergence of t_n will imply convergence of S_k.

Our interest here in this thesis is not in Tauberian theorems of classic type, but rather in studying the manner in which t_n follows S_k, when S_k is the partial sums of a series $\Sigma\, a_k$ satisfying a Tauberian condition. The Tauberian theorems then will be of the following type :-

Suppose that p,n are related in an appropriate way (usually the assumption is that $\frac{p}{n} \longrightarrow \alpha$ as $n \longrightarrow \infty$, where $\alpha > o$ is a constant). Suppose that

(2) $$\limsup_{n\to\infty} |na_n| < \infty \quad ,$$

Then, there is a constant A such that

(3) $$\limsup_{n\to\infty} |t_n - S_p| \leq A \limsup_{n\to\infty} |na_n|$$

The best possible value of the constant A will be determined. However, it was H. Hadwiger who started

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introducing this type of Tauberian theorems, and that was in (1944).

The first part of this thesis is concerned with the history of the researches introduced by many authors for both of the general and special transformations.

The second part of this work is concerned with Meir's theorems. His theorems are analogous to the above mentioned type but when (1) is replaced by a function-to-function transformation. We shall also discuss Soraya Sherif's theorems which are of the above type but when the Tauberian condition (2) is replaced by the weaker condition.

$$(4) \qquad \limsup_{n\to\infty} |Y_n| < \infty \;, \qquad \text{where } Y_n = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu a_\nu .$$

The third part of the work is concerned with Tauberian theorems analogous to the above mentioned type but for the Borel, Hausdorff, and the series-to-series Quasi Hausdorff transformation under different Tauberian conditions.

The Hausdorff and the Borel results where obtained by Biegert. The Quasi Hausdorff results were obtained by Anjaneyulu.

In the third part also, we shall introduced a Tauberian theorem analogous to the above mentioned type but for the Lototsky transformation instead. The result is illustrated as follows :-

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Suppose that

(5) $a_k = O\left(\frac{1}{k^p}\right)$, where p is a fixed real number.

Let m be an integer valued function of n such that

(6) $\limsup_{n\to\infty} |(m - \log n)/(\log n)^k| \leqslant C \qquad (k > 0)$,

where C is a constant. In other words

(7) $m = \log n + O(|\log n|^k) + o\left[(\log n)^k\right]$.

Then,

(8) $\limsup_{n\to\infty} |\sigma_n - S_m| \leqslant \emptyset(C) \limsup_{k\to\infty} |k^p a_k|$,

where $\emptyset(C)$ is a Tauberian constant which will be determined, and σ_n is the Lototsky transform.

We may note that our theorem includes Sherif's Tauberian theorem for the Lototsky transformation which was introduced in (1967), in that our Tauberian condition (5) is general than the condition $a_k = O\left(\frac{1}{k^{1/2}}\right)$.

In the fourth part of this work we shall be concerned with the regularity of the series-to-series Quasi Hausdorff transformation obtained by Ramanujan. We also consider the total regularity results obtained by Dr. Soraya Sherif for the sequence-to-sequence Quasi Hausdorff transformation.

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Again, we shall prove that if

(9) $$\binom{k}{n}(\Delta^{k-n} u_n)s_k \longrightarrow 0 \quad , \text{ as } k \longrightarrow \infty ,$$

then the sequence-to-sequence Quasi-Hausdorff transformation t_n defined by

(10) $$t_n = \sum_{k=n}^{\infty} \binom{k}{n}(\Delta^{k-n} u_{n+1}) \; s_k$$

and the series-to-series Quasi Hausdorff transformation b_n defined by

(11) $$b_n = \sum_{k=n}^{\infty} \binom{k}{n} \; (\Delta^{k-n} u_n) \; a_k$$

are formally the same

Also, we shall prove that (10) and (11) are equivalent. (in the sense that if (10) converges for all n then so does (11), and conversely, and that the sums are then related by

(12) $$t_n = b_0 + b_1 + \dots + b_n \quad , \quad s_k = a_0 + a_1 + \dots + a_k$$

However, these results were introduced by B. Kwee in (1968), but in this thesis we give simple proofs to these o results rather than Kwee's.

We shall also introduce a new Tauberian theorem for the transformation (9) of the above mentioned type. Our proof will be in such a way that we prove the following :-

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If

$$x_1(t) = \int_0^t x(u)\, du = o\left(\frac{t}{\log \frac{1}{t}}\right) ,$$

Then, we have

$$k^n \int_0^1 (1-t)^{k-n}\, t^n\, dx(t) = o\left(\frac{1}{\log k}\right), \text{ as } k \to \infty ,$$

where

$$u_n = \int_0^1 t^n\, dx(t) ;$$

and $x(t)$ is of bounded variation, $x(o) = o$, and $x(1)=1$ and that (9) holds for every series satisfying the Tauberian condition (2), and thus Anjaneyulu's Tauberian theorem of the above mentioned type (theorem 4.1 of Chapter III) for the series-to-series Quasi-Hausdorff transformation b_n applies also to the sequence-to-sequence transformation t_n defined by (10).

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INTRODUCTION

1. Tauberian Theorems

Let $\{S_k\}$ $k \geq o$, $(S_k = a_o + a_1 + \dots + a_k)$ be a sequence of real or complex numbers. Denote by t_k a linear transform T,

$$(1.1) \qquad t_k = \sum_{k=o}^{\infty} C_{n,k} S_k \qquad\qquad n \geq o$$

The transformed sequence t_k can be regarded as a general form of mean of S_k, and there is a familiar principle in analysis that a mean of a sequence generally behaves better than the sequence itself. Thus, for example, it is natural to expecte that, under certain conditions on $C_{n,k}$, the boundedness or convergence as $k \longrightarrow \infty$ of t_k will follow from the same properties of S_k. Theorems of this kind are called abelian theorems after the celebrated theorem of Abel to the effect that the convergence to A of a series $\Sigma\ a_k$ implies that

$$\Sigma\ a_k\ x^k \longrightarrow A \quad \text{as} \quad x \longrightarrow 1 - o\ .$$

The converse theorem, however, that the existence of lim $\Sigma\ a_k\ x^k$ implies the convergence of $\Sigma\ a_k$ is not unconditionaly true. Tauber [48] showed that if

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$$\lim \Sigma a_k x^k = A ,$$

and $$\lim k a_k = o ,$$

it follows that $\Sigma a_k = A$.

Hardy and Littlewood [17] have shown that

$$\lim_{x \to 1-o} \Sigma a_k x^k = A ,$$

and the auxiliary condition that $ka_k > - K$ form a sufficient hypothesis for this condition :

$$\Sigma a_k = A .$$

The term "Tauberian theorem" is applied to theorems of this type, namely those which involve some hypothesis on the rapidity of growth of the terms of a series, and under this assumption proceed in the ordinary sence from the given summability of the series by some particular method to its convergence.

The hypothesis of Tauber's theorem was generalized in some what different direction by Londau in 1913, who proved that the conditions

$$S_k = O(1) ,$$

$$\lim_{\delta \to o} \ \underset{k(1-\delta) \leqslant m \leqslant k(1+\delta)}{\text{maximum}} \left| S_m - S_k \right| = o ,$$

are sufficient. Afterwards, R. Shmidt [38] proved a theorem which includes all these generalizations, in which it is supposed only that

$$\underline{\lim} \; (S_m - S_k) \geqslant o \; ,$$

when $m > k$, and m and k tend to infinity so that $\frac{m}{k} \longrightarrow 1$. Introducing the method of Fourier analysis, Winer [52] brought about not only a systematization, but a simplification of a large portion of the field in the theory of Tauberian theorems. He pute Tauberian theorem in the form: If a function has a mean value of a certain type for large values of its arguments, and if the rate of growth of the function is subject to certain restrictions, then the function has a limiting value of infinity.

In 1938, Pitt introduced a series of general Tauberian theorems which include and extend all those which were most familiar, such as for example Hardy and Littlewood, Schmidt and Wiener.

Several researches has been made afterwards in this type of the field of Tauberian theorems, see for example Agnew (1941) [1] , Sherif [42].

2. Tauberian Constants

In (1944), Hadwiger [14] introduced a new type of Tauberian theorems which involves Tauberian constants. His result was as follows :

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Let $\sigma(t) = \sum_{\nu=0}^{\infty} t^{\nu} a_{\nu} \qquad 0 < t < 1$

and let the condition

(2?1) $$\limsup_{n\to\infty} |n\, a_n| < \infty$$

be satisfied. Suppose that

$$1 - \frac{1}{n} < t_n < 1 - \frac{1}{n+1}$$

Then,

$$\limsup_{n\to\infty} \sigma(t_n) - S_n| \leq \rho \limsup_{n\to\infty} |n\, a_n|$$

where ρ is a constant with value approximately 1.0160. This shows that even though the sequence s_n and the Able transform $\sigma(t)$ may be divergent and perhaps unbounded, we can take a fixed large n and then select a t near 1 such that $\sigma(t)$ lies within a specified distance from S_n; and conversely we can take a fixed t near 1 and then select a large n such that S_n lies within a specified distance from $\sigma(t)$;

The problem actually proposed by Hadwiger was concerned with limit points.

<u>Definition 1</u> A complex number z is a limit point of the sequence S_n of partial sums of $\Sigma\, a_n$ if there is a sequence $n_1, n_2, \ldots$ of integers such that $n_p \to \infty$ and $S(n_p) \to z$. Here and elsewhere we write $S_{(j)}$ for S_j when j is a symbole involving subscripts.

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<u>Definition 2</u> A complex number ζ is a limit point of the transform $\sigma(t)$ of $\Sigma\, a_n$ if there is a sequence $t_1, t_2, \ldots$ such that $t_n \to t_o$ and $\sigma(t_n) \to \zeta$

Hadwiger 14 proved that there is a least constant ρ with the following property:

Let $\Sigma\, a_n$ be a series for which $\limsup_{n\to\infty} |n\, a_n| < \infty$, and then, we have :

<u>Assertion 1</u>. To each limit point z of S_n corresponds at least one limit point ζ of $\sigma(t)$ such that

$$(2.2) \qquad |\zeta - z| \leqslant \rho \limsup_{n\to\infty} |n\, a_n|$$

<u>Assertion 2</u>. To each limit point ζ of $\sigma(t)$ corresponds at least one limit point z of S_n such that (2.2) holds.

Hadwiger showed that $0.4854 \leqslant \rho \leqslant 1.0160$.

The previous results of Hadwiger [14] provide partial motivation for the formulation of problems which were the topics of several investigation stated in our bibliography. The work done on this subject involves general kernels considered in chap: II, as well as special kernels considered in chap. III.

As the bibliography in Sherif [40], [39] several investigations have been done in this topic, when T is a transformation of a more general kind, as well as when it is a special one.

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General Transformations

Rajagopal ([34], [35] Part II and Part III) and also Angew [3], considered transformations of a general form.

Afterwards, in (1963), Meir [30] was concerned with an integral transformation $T(y)$ $(y \geqslant 0)$ of a function $f(x)$ $(-\infty < x < \infty)$ of the form

$$(2.3) \quad T(y) = T_\beta(y) = \int_{-\infty}^{+\infty} f(x)\, d\left[1 - \beta(y-x)\right],$$

where $\beta(x)$ is a function of bounded variation in $(-\infty < x < \infty)$. In section one of chapter II we shall discuss the proof of estimating $\left| T_\beta(y) - T_\gamma(\eta)\right|$, where T_β and T_γ are transforms satisfying certain general conditions, y and η tend to $+\infty$ with a connection on $y - \eta$.

The estimate is follows :

$$\limsup \left| T_\beta(y) - T_\gamma(\eta)\right| \leqslant L.A_q ,$$

where

$$A_q = \int_{-\infty}^{+\infty} \left| \beta(x) - \gamma(x-q) \right| dx .$$

We may note that the Tauberian condition used by Meir was of the simple Hardy form.

In section II, we shall be concerned with similar estimates but for a sequence-to-sequence general transformation, defined as in (1.1).

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