# SPECIAL PROBLEMS IN WALSH-FOURIER SERIES

## THESIS

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# SUMMARY

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## SUMMARY

In recent years, Walsh function theory has been innovated and applied to various fields in engineering and science. Walsh functions were introduced in 1923 by Walsh, J.L. [42]. They form a complete orthonormal system on the interval [0,1) and, although taking only the values +1 and -1, were found to have many properties similar to the trigonometric system.

In this thesis we have studied Walsh functions, Walsh Fourier series, fast Walsh transform, and have established how the Walsh functions and the block-pulse functions can be used in numerical solution of systems of delay and multi-delay differential equations, with initial conditions. The thesis is divided into four chapter.

In chapter 1, we have presented a general introduction through which we have studied the different forms of the Walsh functions ordering, the sequency or Walsh ordering, the Paley ordering and the Hadamard ordering; the dyadic group and its representation on the interval [0,1); Walsh Fourier series; the relationship between Fourier and Walsh series; the operational matrix which relates the Walsh functions and their integrals. At the end of this chapter we have defined the block-pulse functions. The operational matrix which relates the block-pulse functions and their integrals is introduced and the relationship between Walsh operational matrix and block-pulse operational matrix is established.

In chapter 2, we have presented the different forms of the ordinary and fast discrete Walsh Hadamard transform, which are used for the representation of the data sequences. The computation of the discrete Walsh Hadamard transform requires $N^2$ mathematical operations where an operation is either an addition or subtraction, and N is the number of elements in data sequence. But the fast Walsh Hadamard transform requires only $N \log_2 N$ of such operations. We may classify the ordinary and fast finite discrete Walsh Hadamard transform into three types

(1) Fast Hadamard ordered Walsh-Hadamard transform ,
(2) Fast Walsh ordered Walsh-Hardamard transform ,and
(3) Fast Paley ordered Walsh-Hadamard transform.




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In chapter 3, after defining the dyadic derivative and integral, we studied the problem of term by term dyadic differentiation for Walsh series. There are two basic techniques which have been used on this problem, the first technique centers on a calculation which estimates the remainder term and finding the conditions on Walsh coefficients which imply that the remainder term tends to zero at infinity; the second technique depends on finding conditions on Walsh coefficients which an interchange of limit and infinite summation. We illustrated these two techniques in the simple case of lacunary Walsh series. We have studied how Walsh series can be term by term dyadic differentiated. At the end of this chapter we have defined the extension of the dyadic derivative which is not only applicable to piecewise constant functions but also to piecewise polynomial functions of order n having a finite number of discontinuities.

Approximating a function as a linear combination of a set of orthogonal basis functions is a standard tool in numerical analysis. As basis functions in an approximation ,Walsh functions and block-pulse functions lead to the same results. The results are piecewise constant with minimal mean square error.

In chapter 4, we have been concerned with some applications of Walsh and block-pulse functions, where we have established how these functions can be used as basis functions in numerical solutions of systems of delay differential equations with initial conditions. Namely, we have studied the following systems

(1) delay differential equations with variable coefficients
(2) multi-delay differential equations with constant coefficients
(3) multi-delay differential equations with variable coefficients.

Depending on the special properties of the operational matrix for the block-pulse functions, we have established a recursive algorithm to reduce the calculations in each case. The methods described in this chapter are easy to implement on computer. We have desiged computer programs for these methods. These programs have been written in BASIC language and tested on IBM-computer. Outputs of some examples are presented.



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CHAPTER (1)

# INTRODUCTION

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# CHAPTER 1

# INTRODUCTION

## 1.1 Walsh Functions

Walsh functions belong to the class of piece-wise constant basis functions that have been developed in the twentieth century and have played an important role in scientific and engineering applications. The mathematical techniques of studying functions, signals, and systems through series expansions in orthogonal complete sets of basis functions are now standard tools in all branches of science and engineering. The origin of the mathematical study of piece-wise constant basis functions is due to Alfred Haar [19],[20] who used a set of functions bearing his name.These functions have not been of much use in comparison to Walsh and block pulse functions.The foundations of Walsh functions field were developed by Rademacher [34],Walsh [42],Fine [16], [17],Paley [29],Kaczmarz and Steinhaus [25].The engineering approach to the study and utilization of these functions was originated by Harmuth [21]-[23] who introduced the concept of sequency to represent the associated,generalized frequency defined as one half of the average number of zero crossings per unit time interval.The variety of Walsh functions definitons is due to the existence of different orderings. In the sequency ordering (or Walsh ordering), which is popular in communication engineering, Walsh functions are ordered according to the zero crossings( or sign changes ). This sequency ordering implies that the $i^{th}$ Walsh function has $i$ zero crossings in the interval [0,1),(see fig. (1-2)a),and obviously,is directly related to the sequency concept.The Paley ordering [29] is characterized by the fact that in this form Walsh functions are represented by products of Rademacher functions,which lead to useful, recursive, Walsh signal generation algorithms. A third ordering is Hadamard ordering which is merely Paley's ordering in reversed binary. Hadamard's ordering is computationlly attractive and occurs when one computes fast Walsh transform without sorting . Harmuth [21], introduced the Sal and Cal functions which are analogous to the sine and cosine functions. Walsh functions in any ordering form a complete orthonormal system on [0,1),which contains Rademacher system, and enjoys many properties analogous to the classical trigonometric system.



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Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N}$ denote the set of non-negative integers, $\mathbb{Z}$ denote the set of integers, $\mathbb{R}$ denote the set of real numbers, $\mathbb{C}$ denote the set of complex numbers and $\mathbb{Q}$ denote the set of dyadic rationals in the interval $[0,1)$. In particular, any element of $\mathbb{Q}$ can be written in the form $m/2^k$ where $0 \le m < 2^k$ and $k \in \mathbb{N}$. By a dyadic interval in $[0,1)$ we shall always mean an interval of the form

$$I(m,k)=\left[\frac{m}{2^k}, \frac{m+1}{2^k}\right), \qquad 0 \le m < 2^k, \qquad k,m \in \mathbb{N}$$

**Definition (1-1) Rademacher system :[18]**

Let r be the function defined on $[0,1)$ by

$$r(x) = \begin{cases} 1 & x \in [0, 1/2) \\ -1 & x \in [1/2, 1) \end{cases}$$

extended to $\mathbb{R}$ by periodicity of period 1. The Rademacher system $(r_k,\ k \in \mathbb{N})$ is defined by .

$$r_k(x) = r(2^k x) \qquad x \in \mathbb{R}, \quad k \in \mathbb{N}$$

Rademacher functions form an incomplete set of orthonormal functions. It is evident from the definition that

$$r_{k+m}(x) = r_k(2^m x)$$

and

$$\int_{m/2^k}^{(m+1)/2^k} r_k(x)\, dx = 0 \quad , \qquad m,k \in \mathbb{N}$$

It is also clear that each $r_k(x)$ has period $1/2^k$, is constant on the dyadic intervals $I(m,k+1)$and although it has a jump discontinuity at each point of the type $m/2^{k+1}$, it is always continuous from the right. Fig.(1-1) shows the set of the first four Rademacher functions.

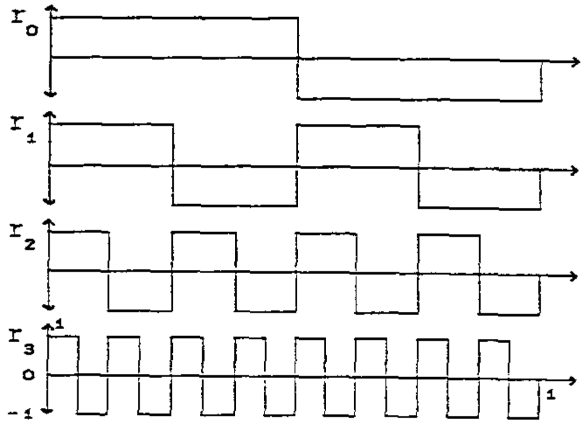


Fig.(1-1) : A set of Rademacher functions



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The incomplete set of Rademacher functions was completed by Walsh [42] in 1923.

**Definition (1-2): Walsh functions in Walsh ordering ($\phi_n, n \in \mathbb{N}$) [30]**

This is the ordering which was originally employed by Walsh [42]. His definition is recursive and takes the following form :

Set $\phi_o = 1$ and $\phi_1 = r_o$.

For each integer $n \geq 2$ choose $m, k \in \mathbb{P}$ such that $1 \leq k \leq 2^{m-1}$ and $n = 2^{m-1} + k - 1$. Set

$$\phi_n = \phi_m^{(k)}$$

where the functions $\phi_m^{(k)}$ are periodic of period 1 and generated recursively by the following process. Define

$$\phi_1^{(1)} = \phi_1$$

and

$$\phi_2^{(k)}(x) = \begin{cases} \phi_1^{(1)}(2x) & , \quad x \in [0, 1/2) \\ (-1)^k \phi_1^{(1)}(2x), & x \in [1/2, 1) \end{cases}$$

for $k = 1,2$. If $m = 2,3, \ldots$ and $1 \leq k \leq 2^{m-1}$ then define

$$\phi_{m+1}^{(2k-1)}(x) = \begin{cases} \phi_m^{(k)}(2x) & , \quad x \in [0, 1/2) \\ (-1)^{k+1} \phi_m^{(k)}(2x), & x \in [1/2, 1) \end{cases}$$

and

$$\phi_{m+1}^{(2k)}(x) = \begin{cases} \phi_m^{(k)}(2x) & x \in [0, 1/2) \\ (-1)^k \phi_m^{(k)}(2x) & x \in [1/2, 1) \end{cases}$$

The Cal and Sal functions corresponding to $\phi_n$, $n \in \mathbb{N}$, may be defined by

$$\left.\begin{aligned} Cal_{i/2}(t) &= \phi_i(t) \quad , i \text{ even} \\ Sal_{(i+1)/2}(t) &= \phi_i(t) \quad , i \text{ odd} \end{aligned}\right\} \qquad (1\text{-}1)$$

The first eight of Walsh functions $\phi_n(t)$ are shown in Fig.(1-2)a

<u>Discrete case [1]</u> : Sampling of Walsh functions in Fig.(1-2)a at eight equidistant points results in the (8x8) matrix shown in Fig.(1-2)b. In general an (NxN) matrix would be obtained. We denote such matrices by $H_w(n)$, $n = \log_2 N$, the subscript "w" denotes Walsh ordering. Let $u_i$ and $v_i$ be the $i^{th}$ bit in the binary representations of the integers U, V, respectively, that is ,



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$$(U)_{decimal} = (u_{n-1}\ u_{n-2}\ \dots\ u_1\ u_0)_{binary}$$

and

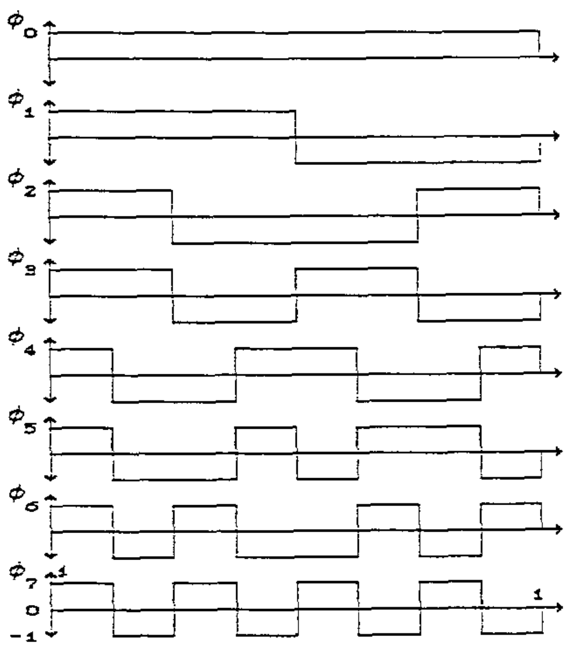
$$(V)_{decimal} = (v_{n-1}\ v_{n-2}\ \dots\ v_1\ v_0)_{binary}$$

Then the elements $h_{uv}^{(W)}$ of $H_W(n)$ can be generated using the relation

$$h_{uv}^{(W)} = (-1)^{\sum_{i=0}^{n-1} r_i(u)\, v_i} \quad , u,\ v = 0,1,2,\dots,N-1 \qquad (1-2)$$

where $r_0(u) = u_{n-1}$ and $r_i(u) = u_{n-i} + u_{n-i-1}$ , $i = 1,2,\dots,n-1$ .

Each Walsh function is piecewise constant with finitely many jump discontinuities on [0,1), and takes only the values +1 or -1




(a) Walsh ordering-continuous Walsh functions , N=8

$$H_W(3) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

(b) Walsh ordering discrete Walsh functions ,N=8

Fig.(1-2)

**Definition (1-3) :**

**Walsh functions in dyadic or Paley ordering** ($W_n$ : $n \in \mathbb{N}$ ) [18]

The dyadic type of ordering was introduced by Paley [29].If $n \in \mathbb{N}$, it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k\, 2^k \qquad (1-3)$$

where $n_k = 0$ or 1 for $k \in \mathbb{N}$. This expression will be called the binary expansion of n and the numbers $n_k$ will be called binary coefficients of n. Walsh function $W_n(x)$ may therefore be defined as products of Rademacher functions in the following way :



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$$W_n(x) = \prod_{k=0}^{\infty} r_k^{n_k} \qquad , n \in \mathbb{N} \qquad (1-4)$$

It is noted that this product is always finite because $n_k=0$ for $k$ sufficiently large. Also by definition, $W_0=1$ and $W_{2^n} = r_n$ for $n \in \mathbb{N}$. Moreover, it is clear that Walsh system is closed under finite products.

For each $0 \le n < 2^{k+1}$, Walsh function $W_n(x)$ is constant on the intervals $[m/2^{k+1}, (m+1)/2^{k+1})$, $0 \le m < 2^{k+1}$ and $W_n(x) = 1$ for all $x \in [0, 1/2^{k+1})$. If $2^k \le n \le 2^{k+1}$, for some $k \ge 0$, then

$$\int_{m/2^k}^{(m+1)/2^k} W_n(x)\, dx = 0 \quad , \quad \text{for each } m = 0,1,2\ldots,2^k-1$$

and

$$\int_0^1 W_n(x)\, dx = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \ge 1 \end{cases} \qquad (1-5)$$

It is immediate from (1-5) that Walsh system satisfies the orthogonality condition,

$$\int_0^1 W_n(x)\, W_m(x)\, dx = \begin{cases} 1 & \text{for } n = m \\ 0 & \text{for } n \ne m \end{cases}$$

i.e., Walsh system forms an orthonormal system on $[0,1)$. By using periodic extension of Walsh functions to the whole real line, it is easy to see that

$$W_{n2^m}(x) = W_n(2^m x)$$

and extend the index set $\mathbb{N}$ to $[0, \infty)$, then

$$W_t(2^k) = W_{2^k}(t) \qquad ,t \in [0,\infty) \qquad (1-6)$$

The first eight of Walsh functions $W_n$ are shown in Fig.(1-3)a.

<u>Discrete case [1]</u> : By sampling Walsh functions in Fig.(1-3)a we obtain the (8x8) matrix shown in Fig.(1-3)b, and in general one would obtain an (NxN) orthogonal and symmetric matrix $H_p(n)$, $n = \log_2 N$, the subscript "p" denotes Paley ordering. The elements $h_{uv}^{(p)}$ of $H_p(n)$ can be generated using the relation

$$h_{uv}^{(p)} = (-1)^{\sum_{i=0}^{n-1} u_{n-i-1} v_i} \qquad , u,v = 0,1,2,\ldots, N-1 \qquad (1-7)$$

and we can see that

$$W_k\left(\frac{j}{2^n}\right) = W_j\left(\frac{k}{2^n}\right) \quad , j,k = 0,1,2,\ldots, N-1 \qquad (1-8)$$



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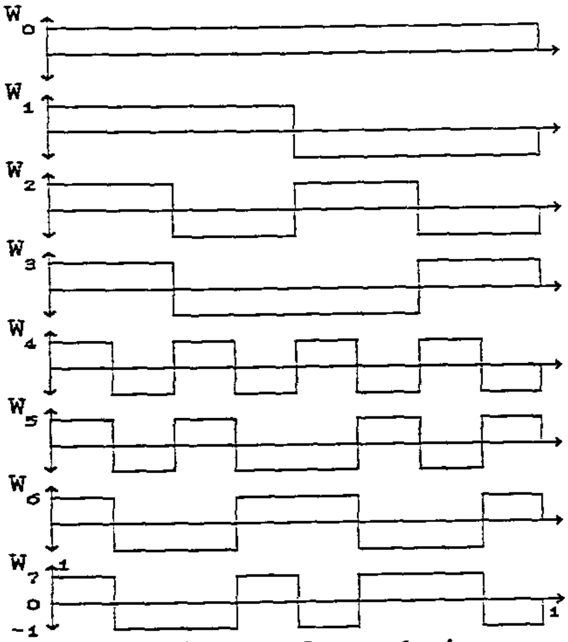


(a) dyadic or Paley ordering, continuous Walsh functions, N=8

$$H_p(3)=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

(b) dyadic or Paley ordering, discrete Walsh functions, N=8

Fig.(1-3)

This set of functions ($W_i$, $i \in \mathbb{N}$ ) is relatd to Walsh ordering functions ($\phi_i$, $i \in \mathbb{N}$ ) by means of the Gray code

$$W_i(t) = \phi_{b(i)}(t)$$

where $b(i)$ is the Gray code of $i$ [4], and if

$$(i)_{decimal} = (b_{n-1}\ b_{n-2} \cdots b_2\ b_1\ b_0)_{binary} ,$$

$$(b(i))_{decimal} = (g_{n-1}\ g_{n-2} \cdots g_2\ g_1\ g_0)_{binary} ,$$

then

$$g_\ell = b_\ell \dotplus b_{\ell+1} , \quad 0 \le \ell \le n-2$$

$$g_{n-1} = b_{n-1}$$ , the symbol $\dotplus$ denotes module 2 addition.

**Definition (1-4): Walsh functions in Hadamard ordering ($\psi_n$, $n\in\mathbb{N}$)[1]**

This set of functions are related to Walsh ordering functions ($\phi_i$, $i \in \mathbb{N}$) by the relation :

$$\psi_i(t) = \phi_{b(\langle i \rangle)}(t)$$

where $\langle i \rangle$ is obtained by the bit-reversal of $i$ and $b(\langle i \rangle)$ is the Gray code of $\langle i \rangle$. They are also related to the Paley ordering Walsh functions ($W_i$, $i \in \mathbb{N}$) by the relation

$$\psi_i(t) = W_{\langle i \rangle}(t)$$

The first eight of Walsh functions $\psi_n$ are shown in Fig.(1-4)a.



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