ON SOME PARTICULAR CLASSES OF INFINITE MATRICES

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(i)	Summability of series	2 hours weekly
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(iv)	Functional analysis	2 hours weekly

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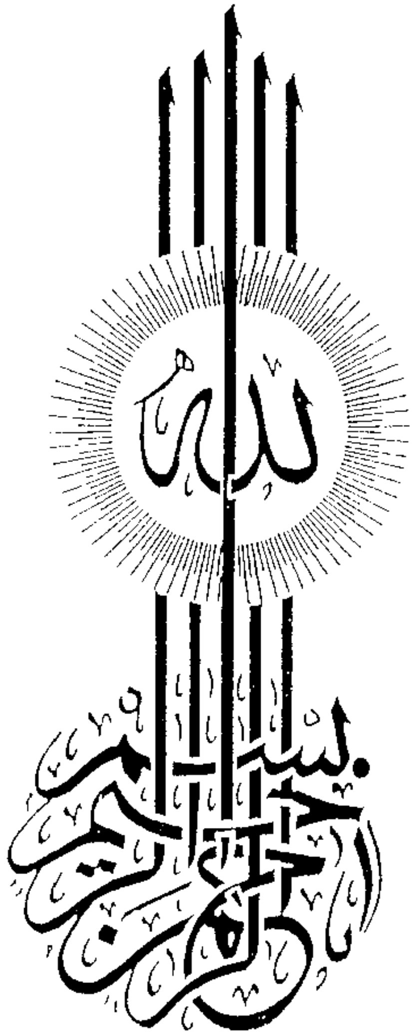

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PREFACE

The thesis deals with the associated matrices of an infinite matrix. By the associated matrices we mean the direct power matrix, the induced matrix and the compound matrix. To simplify the investigation we discuss only the second direct power matrix, the second induced matrix and the second compound matrix though the results given are already extendable to higher cases .

The thesis consists of four chapters. In chapter I we give convenient methods for forming the associated matrices of a string and of a semi-block matrix.

In chapter II we obtain the associated matrices formed by the procedures of chapter I as matrices of transformations.

In chapter III we discuss the eigenvalues , eigenvectors and principal vectors of the associated matrices of a string and of a semi-block matrix.

In chapter IV we investigate the associated matrices of two particular classes of infinite matrices. The first class is that of [illegible]-matrices. The second class

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is the so-called Turnbull diagonal matrices .

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CHAPTER I

ON THE FORMATION OF ASSOCIATED MATRICES OF SOME TYPES OF INFINITE MATRICES

1. On the second direct power matrix of a string

The dirct product of two matrices $A = [a_{ij}]$ and $B = [b_{rs}]$ of orders $m \times n$ and $p \times q$ respectively is defined to be the matrix of order $mp \times nq$

$$A \times B = [a_{ij}\, b_{rs}]$$

of which the element $a_{ij}\, b_{rs}$ is in the (i,r) th row and (j,s) th column, the row and column indexed pairs (i,r) and (j,s) are usually arranged in dictionary order; thus (i,r) precedes (i',r') if $i < i'$ or $i = i'$ but $r < r'$. Indeed according to this definition

$$A \times B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

When $B = A$, the matrix $A \times A$ (of order $m^2 \times n^2$) is written $A^{(2)}$ and is called the second direct power matrix of A. Obviously such a definition cannot be applied to construct $A^{(2)}$ when A is an infinite matrix.

Makar [8,9] has considered an arrangement for the indexed

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pairs (i,r) and (j,s) which enables us to construct $A^{\{2\}}$ when A is an infinite matrix, i.e., we can write down the element $c_{\mu\nu}$ of the matrix $C = A^{\{2\}}$, for any fixed μ and ν. The arrangement considered by Makar is called (by him) the modified dictionary order. This arrangement is described as follows :

(i) permutations involving (two) letters from $1,2,\ldots,i$ precede all permutations involving the letter $i + 1$,

(ii) permutations involving the letter $i+1$ are arranged according to dictionary order ;

(iii) this holds for $i = 1,2,3, \ldots$.

In this arrangement, the permutations appear in the order

$(1,1)$; $(1,2)$, $(2,1)$, $(2,2)$;

$(1,3)$, $(2,3)$, $(3,1)$, $(3,2)$, $(3,3)$;

$(1,4)$, $(2,4)$, $(3,4)$, $(4,1)$, $(4,2)$, $(4,3)$, $(4,4)$; ...

and this arrangement applies to both indexed row pairs and indexed column pairs. It is clear that the number of permutations involving (two) letters from $1,2, \ldots, n$ in this arrangement is

$$\sum_{r=1}^{n} (2r-1) = n^2 .$$

Actually the method constructs successively the 2nd power matrices of the leading submatrices $A_1, A_2, A_3, \ldots$, of A, of orders $1,2,3, \ldots$ respectively. It is readily seen that according to this definition of $A^{\{2\}}$, when A is an infinite diagonal matrix, $A^{\{2\}}$ is also an infinite diagonal matrix .

Indeed when

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$$A = \operatorname{diag}\{a_1, a_2, a_3, \ldots\}$$

we get

$$A^{\{2\}} = \operatorname{diag}\{a_1^2, a_1a_2, a_2a_1, a_2^2, a_1a_3, a_2a_3, a_3a_1, a_3a_2, a_3^2, \ldots\} .$$

It is interesting to consider the structure (shape) of the 2 nd direct power matrix of a given infinite matrix of some certain shape. We start with what is called a string (an infinite quasi - diagonal matrix) . The general phenomena is easily recognised, and formulated, by considering illustrative examples .

We consider the string which starts with the three beads* of sizes, 3,2,2 respectively. Thus the matrix A is of the shape

$$(1.1) \quad A = \begin{bmatrix} x & x & x & 0 & 0 & 0 & 0 \\ x & x & x & 0 & 0 & 0 & 0 \\ x & x & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & x & 0 & 0 \\ 0 & 0 & 0 & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x \end{bmatrix} .$$

Forming the matrix $B = A^{(2)}$ by applying the above rule

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* The names " string and bead " are due to Vermes [12]

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of the modified dictionary order we see that the leading submatrix of order 49x49 of B (and which is actually the 2 nd direct power matrix of the leading submatrix of order 7 of A) can be written in the partitioned form

$$(1.2)\quad B = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{bmatrix}$$

where B_1, B_2 , B_3 are square submatrices of orders 9, 16 and 24 respectively. The submatrix B_1 is a full matrix (i.e.a matrix of which all the elements might be non-zero elements), and its rows as well as its columns correspond to the indexed pairs,

(i) (1,1); (1,2), (2,1), (2;2); (1,3), (2,3), (3,1),(3,2),(3,3).

The rows and columns of B_2 carry the indexed pairs (in the whole matrix B) from (1,4) to (5,5) and the rows and columns of B_3 carry the indexed pairs (in the whole matrix) from (1,6) to (7,7) .

The construction of the leading submatrix of order 49 indicated above clarifies the following result :

When A is a string = diag (A_1, A_2, A_3, ...)

where the bands A_1, A_2, A_3, ... are respectively of orders

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$r_1, r_2, r_3, \ldots$, the second direct power matrix $B = A^{\{2\}}$, constructed according to the modified dictionary order, is a string $B = \text{diag}\,(B_1, B_2, B_3, \ldots)$ wher the beads $B_1, B_2, B_3, \ldots$ are respectively of orders

$$r_1^2 ,\ (r_1 + r_2)^2 - r_1^2 ,\ (r_1 + r_2 + r_3)^2 - (r_1 + r_2)^2 , \ldots .$$

This result can be simply proved by mere inspection and the above example is a complete illustration . The orders of the beads can also be written as

$$r_1^2 ,\ r_2^2 + 2\, r_1 r_2 ,\ r_3^2 + 2\, r_1 r_3 + 2\, r_2 r_3 , \ldots \quad .$$

The construction of each of B_2 and B_3 in (1.2) is worthy of inspection . Indeed B_2 is of the shape in (1.3) . There is a clear isolation of submatrices . Indeed the following interchanges of rows and corresponding columns in B_2 or in the whole matrix $B = A^{\{2\}}$

$$(4,1) \longleftrightarrow (1,5),\ (4,2) \longleftrightarrow (2,5),\ (4,3) \longleftrightarrow (3,5)$$

isolate (in B_2) a leading submatrix of order 6 . This submatrix corresponds to the indexed pairs

(ii) (1,4), (2,4), (3,4) , (1,5), (2,5),(3,5).

Following this by a further sequence of interchanges which makes the permutations

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(iii) (4,1), (4,2) , (4,3) , (5,1) , (5,2) , (5,3)

(1.3)

$$
\begin{array}{c}
(1,4) \\ (2,4) \\ (3,4) \\ (4,1) \\ (4,2) \\ (4,3) \\ (4,4) \\ (1,5) \\ (2,5) \\ (3,5) \\ (4,5) \\ (5,1) \\ (5,2) \\ (5,3) \\ (5,4) \\ (5,5)
\end{array}
\left[
\begin{array}{cccccccccccccccc}
x & x & x & . & . & . & . & x & x & x & . & . & . & . & . & . \\
x & x & x & . & . & . & . & x & x & x & . & . & . & . & . & . \\
x & x & x & . & . & . & . & x & x & x & . & . & . & . & . & . \\
. & . & . & x & x & x & . & . & . & . & . & x & x & x & . & . \\
. & . & . & x & x & x & . & . & . & . & . & x & x & x & . & . \\
. & . & . & x & x & x & . & . & . & . & . & x & x & x & . & . \\
. & . & . & . & . & . & x & . & . & . & x & . & . & . & x & x \\
x & x & x & . & . & . & . & x & x & x & . & . & . & . & . & . \\
x & x & x & . & . & . & . & x & x & x & . & . & . & . & . & . \\
x & x & x & . & . & . & . & x & x & x & . & . & . & . & . & . \\
. & . & . & . & . & . & x & . & . & . & x & . & . & . & x & x \\
. & . & . & x & x & x & . & . & . & . & . & x & x & x & . & . \\
. & . & . & x & x & x & . & . & . & . & . & x & x & x & . & . \\
. & . & . & x & x & x & . & . & . & . & . & x & x & x & . & . \\
. & . & . & . & . & . & x & . & . & . & x & . & . & . & x & x \\
. & . & . & . & . & . & x & . & . & . & x & . & . & . & x & x
\end{array}
\right]
$$

come next, another submatrix of order [illegible] is isolated. The rows and columns corresponding to the remaining four permutations may be taken in the order

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(iv) (4,4) , (4,5) , (5,4) , (5,5) .

This means that if in the construction of $A^{(2)}$, we replace the arrangement of 16 indexed pairs indicated in (1.3) by the arrangement in (ii), (iii) , (iv) taken in order, in place of B_2 shown in (1.3) we get a quasi - diagonal matrix whose (diagonal) blocks are respectively of orders 6, 6 and 4 .

The submatrix B_3 is found to be of the shape in (1.4) .

Again there is a clear separation of submatrices . An arrangement of the inde xed pairs which makes the permutations
(v) (1,6), (2,6) , (3,6) , (1,7), (2,7) , (3,7)
come first isolates a leading submatrix of order 6 . A further arrangement which makes the permutations
(vi) (4,6) , (5,6) , (4,7) , (5,7)
comes next isolates a further submatrix of order 4 . Further arrangements which make the permutations
(vii) (6,1), (6,2), (6,3), (7,1), (7,2), (7,3)
(viii) (6,4) , (6,5) , (7,4) , (7,5)
(ix) (6,6) , (6,7) , (7,6) , (7,7)
come next in this order isolate, in order, submatrices of orders 6,4,4 respectively. Thus if the arrangement, of permutations in (1.4) , from (1,1) to (7,7), is replaced by the arrangement indicated in (v) to (ix) in order, in place

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of the block B_3 of order 24, we get a quasi - diagonal matrix of which the (diagonal) blocks are of orders 6,4,6,4,4 and these correspond to the sets of permutations in (v) to (ix) respectively .

The above argument makes it clear that when we are

$$
(1.4)\quad
\begin{array}{c|cccccccccccccccccccccccc|}
(1,6) & x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\
(2,6) & x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\
(3,6) & x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\
(4,6) & \cdot&\cdot&\cdot&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\
(5,6) & \cdot&\cdot&\cdot&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\
(6,1) & \cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot \\
(6,2) & \cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot \\
(6,3) & \cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot \\
(6,4) & \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&\cdot&\cdot \\
(6,5) & \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&\cdot&\cdot \\
(6,6) & \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&\cdot&\cdot&\cdot&\cdot&\cdot&x&\cdot&\cdot&\cdot&\cdot&\cdot&x&x \\
(1,7) & x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\
(2,7) & x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\
(3,7) & x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\
(4,7) & \cdot&\cdot&\cdot&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\
(5,7) & \cdot&\cdot&\cdot&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\
(6,7) & \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&\cdot&\cdot&\cdot&\cdot&\cdot&x&\cdot&\cdot&\cdot&\cdot&\cdot&x&x \\
(7,1) & \cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot \\
(7,2) & \cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot \\
(7,3) & \cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&x&\cdot&\cdot&\cdot&\cdot \\
(7,4) & \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&\cdot&\cdot \\
(7,5) & \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&x&\cdot&\cdot \\
(7,6) & \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&\cdot&\cdot&\cdot&\cdot&\cdot&x&\cdot&\cdot&\cdot&\cdot&\cdot&x&x \\
(7,7) & \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&x&\cdot&\cdot&\cdot&\cdot&\cdot&x&\cdot&\cdot&\cdot&\cdot&\cdot&x&x
\end{array}
$$

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Considering strings, Makar's procedure, of arranging the indexed pairs of rows and columns according to the modified dictionary order, is not the best one for constructing an $A^{\{2\}}$. Other procedures enable us to produce some $A^{\{2\}}$ in more simple forms (and as we shall see enable us to write down these $A^{\{2\}}$ much more simply) .

Inspection of the arrangement of the 49 permutations (1,1) to (7,7) in the subsequent sets in (i) to (ix) and inspection of the decomposition of the submatrices B_2 and B_3 by the sub - arrangements in (ii) to (iv) and in (v) to (ix) respectively show that when A is a string the new second direct power matrix is obtained by applying the following procedure .

We write A as

$$(1.5) \qquad A = \begin{bmatrix} A_1 & 0 & 0 & \cdots \\ 0 & A_2 & 0 & \cdots \\ 0 & 0 & A_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} .$$

<u>The new second direct power matrix is the string whose beads are the direct product matrices $A_i x A_j$, i= 1,2,3,... , j = 1,2,3, ... arranged according to the modified dictionary</u>

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