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DIFFERENTIAL OPERATORS IN SOME SPACES

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THIS THESIS

ON

DIFFERENTIAL OPERATORS IN SOME SPACES

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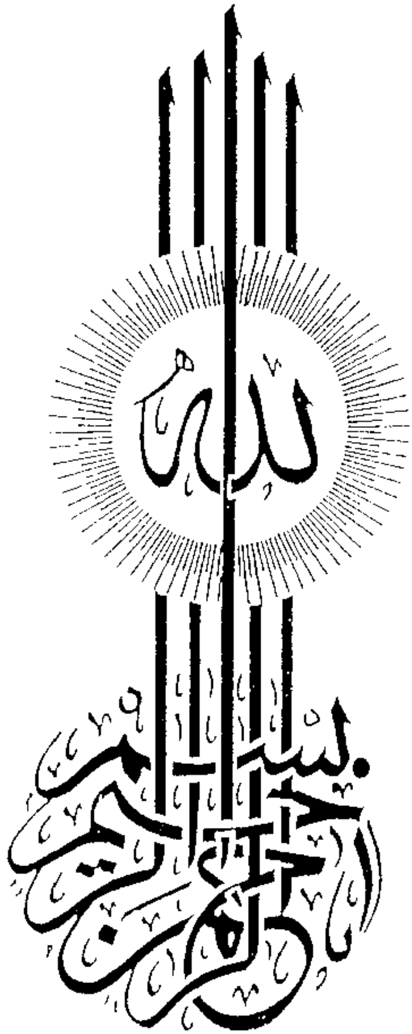
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STATEMENT

This dissertation is submitted to Ain Shams University for the Degree of Master Science in Engineering Mathematics.

The work included in this thesis was carried out by the author in the Depatment of Engineering Physics and Mathematics, Ain Shams University, from 1990 to 1994.

No part of this thesis has been submitted for a degree or a qualification at any other University or Institution.

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ABSTRACT

This thesis presents some results in the area of differential operators on some generalized function spaces (Sobolev spaces) , more precisely , we study some differential operators (Cauchy problems) , then, the aim of the work is to study spectral theory of operators and variational methods.

This thesis is divided into four chapters , chapter 0 is concerned with fundamental concepts and important definitions for revision to study the more elementary properties that are concerned with some of the fundamentals of functional analysis .

Chapter 1 is concerned with studying in details different types of operators , especially differential operators with constant coefficient , then, studying the distributions , self - adjoint operators , and Sobolev spaces. Moreover , we study fundamental solution in order to study some differential operators (Cauchy problems).

While chapter 2 is concerned with studying the spectral theory of operators and their effect in some operators , especially , the spectrum of a self-adjoint operators , then, studying the decomposition theory and limit points of the spectrum .

Finally , we are interested in solving certain one dimensional Schroedinger operator of Sturm Liouville type with a discontinuous coefficient on the half - axis $[0,\infty)$

$$- y'' + e^{-\delta x^{\alpha}} y = \lambda \rho(x) y \quad , \quad 0 < x < \infty \qquad y' \cong 0 ,$$

$$0 < \alpha < \frac{1}{2} \quad \delta > 0 ,$$

by using perturbations , and concluding the two conditions :-

$$\frac{\delta^2 A}{\delta x^2} + \frac{\delta^2 A}{\delta t^2} = - (ka)^2 \ A(x,t)$$

and
$$\frac{\delta^2 B}{\delta x^2} + \frac{\delta^2 B}{\delta t^2} = - (ka)^2 \ B(x,t)$$

Moreover , chapter 3 is concerned with studying variational methods, especially, the Ritz method and Galerkin's method , and then, solving some applied examples to make a comparison between the two methods , and concluding that the two methods are nearly equivalent.

Finally, deducing two improved formulae for both Ritz and Galerkin methods in order to simplify calculations for defining the coefficients that concerning the two methods , and solving one applied example for the two improved methods.

CONTENTS

CHAPTER 3 : VARIATIONAL METHODS

CHAPTER (0)

FUNDAMENTAL CONCEPTS AND THEOREMS

CHAPTER (0)

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FUNDAMENTAL CONCEPTS AND THEOREMS

Introduction

The principal aim of this chapter is to study elementary properties which are concerned with some of the fundamentals of functional analysis , and some main definitions[9] and also , discussing the topological linear spaces [1] , which can be used to give a great background of the importance of a Hilbert space[5] ,which is the greatest successes of the abstract approach to linear problems. A Hilbert space is a very special type of linear space with a topology , and it has many of the properties of Euclidean space . Also, Hilbert space can be viewed as arising from a natural generalization of properties of finite dimensional Euclidean spaces , while Hilbert's initial investigations leading to the class of spaces normed for him which were not exactly based on this fact. The most famous operator which would be studied is the differential operator.

Operations of a special type , are called projections [5] play an important role in the systematic study of linear operators, where the notion of a projection is closely related to the concept of a direct sum of linear manifolds.

0.1 Definitions

A space is a collection of elements, together with a certain structure of relations between elements or combination.

A linear space

Let X be a set of elements $x, y, \ldots$ & let α & β be any real numbers . Then the set X is called **a linear space** if the following axioms are satisfied :-

1. $x + y = y + x$
2. $x + (y + z) = (x + y) + z$
3. $x + 0 = x$
4. $x + (-x) = 0$ for each x
5. $\alpha(x + y) = \alpha x + \alpha y$
6. $(\alpha + \beta)x = \alpha x + \beta x$
7. $\alpha(\beta x) = (\alpha \beta)x$
8. $1.x = x$
9. $0.x = 0$

Noting that elements of X are called **vectors** , and the elements α , β are called **scalars**.

Direct Sum Decomposition

The subspaces $\{ M_i \}$ are said to form **a direct sum decomposition** for X , such that :-

$$X = M_1 \oplus M_2 \oplus \ldots \oplus M_n$$

i.e. every vector in X can be written **uniquely** in the form

$$x_1 + x_2 + \ldots + x_n , \quad \text{where,} \quad x_j \in M_j.$$

A basis

The set S is called **a basis** for the space X if it is the maximal linear independent set in X , and it generates the whole X.

Dimension of a space

It is said that a linear space X is of **finite dimension** n if it has basis consisting of n-vectors. Otherwise, the space is of **infinite dimension**.

An Operator

An operator is a mathematical rule which when applied to a function produces another function.

For example :-
$$L\,[U] = \frac{\delta^2 U}{\delta x^2} + \frac{\delta^2 U}{\delta y^2}$$

hence,
$$L = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}$$

is called a **differential operator**.

Linear operators $T : V \longrightarrow W$ -

A linear operator is a certain kind of function whose domain is a linear space and whose range is contained in a linear space. Condition of a linear operator :-

$$T\,(a\,v + b\,u) = a\,T\,(v) + b\,T(u)$$

OR $T(av) = a\,T(v)$, $T(av) = a\,T\,(v)$

Noting that, if $f : V \longrightarrow R$ then, the linear operators is called **linear functionals**.

The Null Space N(T)

The null space N(T) of T is the set of all elements of the domain of whose image is zero such that :-

$$N(T) = \{\, u \in U \;:\; Tu = 0 \,\}.$$

Algebraic conjugate of X

Let X be a linear space . Let X^f be the class of all linear functionals on X. Then X^f becomes a linear space and is called algebraic conjugate of X. Noting that X^f plays an important role in the study of linear operators with domain X.

Extension

If X is a set , M is a proper subset of X , f is a function defined on M , then , a function F defined on X is called an extension of f if :-

$$F(x) = f(x) \qquad \text{when} \qquad x \in M$$

Noting that the extension F has certain properties possessed by f such that bounded, continuous , differentiable , etc.

The transpose of a linear operator

Suppose X & Y are linear spaces, let A be a linear operator on X into Y , then, for every $y' \in Y^f$ there exists $x' \in X^f$ such that:-

$$\langle x , x' \rangle = \langle A x , y' \rangle$$

Defining a function by A^T such that $A^T y' = x'$ where A^T is a linear operator on Y^f into X^f. and may be written in the form :-

$$\langle x , A^T y' \rangle = \langle Ax , y' \rangle \qquad x \in X \quad , \quad y' \in Y^f$$

$A^T \equiv$ transpose of a linear operator A

Topological spaces

A set X with a family $\mathcal{F}$ of subsets is called a topological space if $\mathcal{F}$ satisfies the following properties :-

1. The empty set Φ and the whole space X belong to $\mathcal{F}$.
2. The union of any number of members of $\mathcal{F}$ is a member of $\mathcal{F}$.
3. The intersection of any finite number of members of $\mathcal{F}$ is a member of $\mathcal{F}$.

Then, the family $\mathcal{F}$ is called a topology for X , and the members of $\mathcal{F}$ are called the open sets of X in this topology.

Separable Space

A topological space X is called separable if there exists a finite or countable set S which is dense in X.

Analytic Function

A function $f(x)$, $x = (x_1,\dots,x_n)$ is called analytic at a point x_o if in a neighborhood of this point x_o , it can be represented by a uniformly converging power series :-

$$f(x) = \sum_{|\alpha|\geq 0} C_\alpha (x-x_o)^\alpha = \sum_{|\alpha|\geq 0} \frac{D^\alpha f(x_o)}{\alpha !} (x-x_o)^\alpha$$

where, the point x_o may lie in the complex plane.

Noting that,if the function $f(x)$ is analytic at each point of a region G, then the function is said to be analytic in G.