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Geometry Of Banach Spaces
And
Operator Ideals

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INTRODUCTION

INTRODUCTION

Functional analysis is occupying one of the central positions in contemporary mathematics . The purpose of this thesis is to give a survey on geometry of Banach spaces and operator ideals . The progress in the theory of geometry of Banach spaces often depends on the discovery of new results in the geometry of Euclidean spaces . This is due to the fact that many problems in the theory of Banach spaces may be reduced to the finite-dimensional case .

In 1961 A.Dvoretzky [10] proved that, in any infinite dimensional Banach space, there exist subspaces B of arbitrary large dimension which are nearly isometric to the Euclidean space l_2^n i.e. the Banach-Mazur distance $d(B,l_2^n)$ between B and the Euclidean space l_2^n is nearly one . This profound theorem makes essential use of an important theorem on ε-isometry of Banach spaces and hinges on the theory of symmetric convex bodies in Euclidean spaces .

There are many notions associated to geometry of Banach spaces such as the existence of a basis . In 1927 J.Schauder posed the fundamental question, about whether every separable space has a basis . This problem remained open for a long time and was solved in the negative by P.Enflo [11] in 1973. In the same time A.Grothendieck has defined the approximation property and raised the question, whether every separable Banach space has the approximation property. In fact, P.Enflo had constructed a subspace of c_o which does not have the bounded approximation property.

In this thesis, we mention a modified version of Enflo's solution due to A. Szankowski [44] . In 1972 Morrell and Retherford [29], using A.Doveretzky's result and the work of V.I.Gurrari [17] about inclinations, had constructed a basic sequence possessing supplementary good conditions which are useful in constructing some operators (see3.4).

We study some types of Banach spaces such as :

1) Reflexive Banach spaces like l_p, $1 \le p < \infty$ spaces which are characterized by having their unit balls weakly compact.

2) Smooth, uniformly smooth, strictly convex, uniformly convex spaces and the duality between the notions of uniform convexity and uniform smoothness .

3) Another type was Banach spaces with basis especially Banach spaces with symmetric basis where the unit vector basis of c_0 and l_p, $1 \le p < \infty$ besides being unconditional, is equivalent to any of its permutations .

In1977 J. Lindenstrauss and L. Tzafriri [24] asked whether there is a Banach space with, up to equivalence, more than one symmetric basis. In 1981 C. J. Read, [36] answered the question positively, by exhibiting a Banach space with, up to equivalence, precisely two symmetric bases.

The idea of ideals of operators has been founded and was apparent in the study of tensor product in Schatten's as well as in Grothendieck's work in the early fifties. In 1968 A.Pietsch and his school began a systematic investigation of operator ideals on the class of Banach spaces and developed a method of thinking in a "categorical" manner which is just as powerful as thinking in terms of tensor products. This development culminated in 1978 in the publication of Pietsch's book "Operator Ideals" which contains in a nearly encyclopedic way everything that was known about operator ideals at the time. Although many of the ideas and results clearly came from the "Résuméé", tensor products were never used in his book.

In this thesis we study some ideals of operators in Banach spaces such as absolutely summing operators, completely continuous operators, nuclear operators and p-Schatten Von Neumann operators. We study the work of [27] which gives a general way of constructing a compact operator of the form $\sum_{i\in N} \lambda_i f_i \otimes z_i$ in arbitrary infinite dimensional Banach spaces such that their sequence of approximation numbers deliberately lies between certain given upper and lower bounds. . This construction helped answering the problem positively that the ideal of p-Schatten Von Neumann operators is a small "distinguishing" ideal. In [33] was given exact estimations of the approximation numbers of any multiplication operator $T = \{\tau_i\}_{i\in I}$ defined on the space $l_\infty(I)$ of bounded families of real numbers. In [12] and [13] were given upper and lower estimations of the approximation numbers of an infinite matrix $T = \{\tau_{nk}\}_{n=1\,k=1}^{\infty\ \ \infty}$ on the space l_1 and l_∞ respectively. In this direction we give upper and lower estimations of the approximation numbers of an infinite matrix with uniformly summable rows and uniformly summable columns. Such type of matrices define bounded linear operators from any sequence space l_p, $1 \le p \le \infty$ into itself. This result is compatible with that given in [12]and[13].

SUMMARY

This thesis consists of four chapters .

Chapter 0 :

In this chapter we give a short introduction to the working tools from functional analysis which are needed for the development of different topics of this thesis .

Chapter I :

In this chapter we study basic definitions and properties of some types of Banach spaces such as : The reflexive Banach spaces, the criterion for reflexivity, the principle of

local reflexivity, strictly convex, uniformly convex, smooth, uniformly smooth and superreflexive Banach spaces . The moduli of convexity and of smoothness and duality between the notions of uniform convexity and uniform smoothness . A useful criterion for checking whether a given sequence is a basis for X . At the end of this chapter, we consider the work of C. J. Read, [34] where he exhibited a Banach space with, up to equivalence, precisely two symmetric bases .

Chapter II :

This chapter is devoted to studying some properties of basic sequences in Banach spaces . We consider the work of Gurrari [17] in which a conditional basic sequence is constructed and the work of Morrell and Retherford in which they had constructed a basic sequence possessing supplementary good conditions which are useful in constructing some operators. The existence of basic sequences with small basis constant by using the principle of local reflexivity due to the work of Catherine Finet [16].

The first example of a Banach space which does not have the approximation property, was given by P. Enflo [11] . We shall present in section four a modified version of Enflo's solution which is due to A. Szankowski [44] .

Chapter III

In this chapter we study some ideals of operators in Banach spaces such as the ideal of absolutely summing operators, completely continuous operators and nuclear operators . We study the work of [13] in which he helped answering the problem positively that the ideal of p-Schatten Von Neumann operators is a small "distinguishing" ideal. By the end of this chapter we give upper and lower estimations of the approximation numbers of an infinite matrix operator on the spaces l_p, $1 \le p \le \infty$.

CHAPTER 0

Chapter 0
Preliminary Concepts

In this chapter we present some basic definitions and facts from functional analysis . These preliminaries will be used in the subsequent chapters.

0.1. Notations

For two Banach spaces X and Y, by L(X,Y), we denote the space of all bounded linear operators from a normed space X into a normed space Y .

$l_\infty(I) = \left\{ x = (x_i)_{i\in I} \ : \ \sup_i |x_i| < \infty \right\}$ denotes the space of all bounded families of real or complex numbers, equipped with the norm $\|x\| = \sup_i |x_i|$. In case for I is the set N of natural numbers, we denote it by $l_\infty(N) = l_\infty = \left\{ x = (x_i)_{i=1}^{\infty} \ : \ \sup_i |x_i| < \infty \right\}$.

$l_p = \left\{ x = (x_i) : \ \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$; $1 \le p < \infty$, denotes the space of all p-summable sequences of real numbers equipped with the norm $\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$.

l_p^n denotes the space of n- tuples with the above norm .

c_0 denotes the space of all sequences converging to zero.

$\Im(X,Y)$ denotes the space of all finite dimensional operators from X into Y .

card $\wp$ denotes the number of elements of a subset $\wp$.

N(X,Y) denotes the space of all nuclear operators from the normed space X into the normed space Y .

If X is a Banach space : We shall denote by

$X^* = L(X,R)$ its dual (conjugate) space,

I_X the identity operator of X,

$$X = F \oplus M,$$

(ii) There is a bounded projection $Q : X \to M$ with

$$\|Q\| \le 1 + \|P\| .$$

0.2.29.**Definition** : [45]

A linear subspace F of a linear space X is said to have **finite codimension n** $(1 \le n < \infty)$ in X if and only if there exists, n linearly independent vectors $z_1, z_2, \dots, z_n$ not belonging to F such that every element $x \in X$ is written uniquely in the form

$x = y + \sum_{i=1}^{n} \lambda_i z_i, \quad y \in F$ and λ_i ; $i = 1,2,\dots,n$ are scalars .

0.2.30.**Definition** : [4] (Quotient space)

Let X be a linear space and M be a subspace of X . Then the relation $(\approx)$ defined by:

$x \approx y$ for $x, y \in X$, if and only if $x - y \in M$ where $x - y = x + (-y)$. The relation is an equivalence relation and therefore divides X into disjoint equivalence classes . Denote the set of all equivalence classes by X / M .

The map $x \to x + M$ which orders to each x its equivalence class x+M is called the canonical (or **quotient) map** of X onto X/M .

0.2.31.**The Converse of Holder Inequality:**

If $\left|\sum_{i=1}^{\infty} a_i b_i\right| \le c$ for some constant c and for every (a_i) in the unit ball of l_p then $\left(\sum_{i=1}^{\infty} |b_i|^q\right)^{1/q} \le c$ for $1/p + 1/q = 1$.