ON SOME APPLICATIONS OF GREEN'S FUNCTION

THESIS

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ABSTRACT

The application of the method of Green's function to hydrodynamic problems is revised in order to outline the method and techniques used for the propagation of waves and its diffraction by head lands and off shore islands. Following the derivation of the basic equations of motion, Green's function and its complex conjugate are determined consistent with the basic equations and derived for the upper-half plane. This approach was originated by Davies (1974) and is critically examined and modified here. A similar Greene's function was derived recently by Voit and Sebekin (1970) but their work did not consider the conjugate function. The complex conjugate is necessary when considering diffraction by an island since the circulation around the island introduces multivalnedness - incorporated in the conjugate function.

Two specific applications of the method are considered in this thesis to calculate the diffraction of Kelvin waves for, namely, semi-circular headland and a circular off-shore island, respectively. These problems are taken up to varying stages of mathematical analysis.

CHAPTER I

INTRODUCTION

INTRODUCTION:

The theory of linear systems is a widely studied branch of Applied Mathematics and theoretical physics. Possibly the most important aspect of the study has been that concerned with Green's Function of the system. Non linear effects, if present, are generally small compared with linear terms and are usually treated as perturbations of the basic linear system. It is found that Green's function is usually an analytic function of frequency and that its mathematical properties are related in a simple way to the physics of the system being described (Webb, 1974).

Now-a-days some of the methods that have been found useful in the study of the other linéar systems are applied to the study of tides. By using these methods, the theory of the tides can be developed in a formal manner. This formal theory gives insight into the physics of the ocean, and it can be extended to develop useful approximations. In particular one can obtain a set of a simple equations for tidal prediction.

It is the object of the present thesis to give, if possible, a wide survey of the application of Green's function in the theory of tides and try to select some particular problems in the field where we may develop the theory. Munk and Cartwright (1966) and Cartwright (1968) succeeded in calculating the response functions, Green's function for the tides, for Hanolulu with a resolution of one cycle per year. They found that the response functions were smooth functions of frequency, except for a small jitter that they persumed was due to some weak nonlinearity in the tide.

By hypothesizing that the response functions were smooth, Munk and Cartioright (1966) developed a truncated fourier series technique that could be readily extended to include nonlinear terms. This was successfully used to analyse the tides at Newlyn. Cartwright (1968) made similar calculations for a number of other British ports, and the technique was also used, without the nonlinear terms, by Fliegel and Nowroozi (1970), Munk et al (1970), Fillaux (1971) and Cartwright (1971). A slightly different approach was used by Wunsch (1972). The success of these papers supports the hypothesis that the tidal response functions are usually smooth functions of frequency.

Webb (1974) outlined some of the more theoretical aspect of applying the Green's function theory to the ocean. In his detailed paper he established the relashonships between the mathematics of the Green's function and the physics of the ocean, he calculated the equations for the tide, and these are used to obtain expressions for predicting the tide. He showed that the response function can be split into a resonance part and background part, and that the semidiurnal response function at Cairns indicates a resonance of the Coral Sea superimposed on a smooth background. Finally, he outlined that these expressions may be suitably approximated to obtain a class of very simple formulae that are useful in tidal prediction.

In chapter two we consider Green's function for the Helmholtz equation and prove the symmetry properties of the function. The relation between Green's function and the boundary conditions and the density function is illustrated by some examples. This chapter also contains some different forms of Green's function which correspond to some kinds of differential operators.

Chapter three contains a fairly derivation of the governing equations of motion and the wave equation for the tidal motion. This is followed by short sections on Kelvin Wave system, and forced oscillation along a gulf.

Chapter four contains the derivation of Green's function and its conjugate arising out of classical tidal theory [Davies,(1974)].

Chapter five, which is considered as the main part of the thesis, consists of idealised problems, all of which have been analysed up to varying points of mathematical development. We consider in turn the diffraction of an incident Kelvin Wave by a narrow off shore island, by a semicircular headland and by a circular off-shore island. In each case, motion takes place in a semi-infinite sea, bounded by a straight coast, and the Kelvin Wave is directed parallel to the coast.

The problem involving a semi-circular headland and a circular off-shore island have been examined from the first principles but have not been taken as far as the problem of the narrow headland. In the case of the narrow off-shore island problem, the source distribution has been determined and the solution for the tidal height can be given in closed integral form. The semi-circular headland and the circular off-shore island problems have been taken as far as the determination of the singular integral equation.

CHAPTER II

THEORY OF GREEN'S FUNCTION

2.1. Introduction

In finding solutions of a given partial differential equation, subject to suitable boundary conditions, it is disirable to have a closed form function representing such solutions. If it is an integral representation, the Green's function technique is just such an approuch.

To obtain the field caused by a distributed source we calculated the effects of each elementary portion of source and add them all. If G (r/r_o) is the field at the observer's point r caused by unit point source at the source point r_o , then the field at r caused by a source distribution $\rho(r_o)$ is the integral of $G\rho$ over the whole range of r_o occupied by the source. The function G is called the Green's function.

We compute the field at r for the boundary value zero at every point on the surface except for r_o^s (which is on the surface). At r_o^s the boundary value has a delta function behavior, so that its integral over a small surface area near r_o^s is unity. This field at r (not on the boundary) we can call G (r/r_o^s), then the general solution, for an arbitrary choice of boundary values ψ_o (r_o^s) is equal to integral of $G\psi_o$ over the boundary surface, these functions are also called Green's functions.

The Green's function is therefore a solution for a case which is homogeneous everywhere except at one point. When the point is on the boundary, the Green's function may be used to satisfy inhomogeneous boundary concintions, when it is out in space, it may be used to satisfy the inhomogeneous equation.

Green's function depends on the time characteristic of the source and it can be either harmonic e^{iwt} or impulsive $\delta(t)$.

Depending on the physics of the system, under consideration, the Green's function can be given other more descriptive names. In tidal analysis, it is becoming accepted [Webb, 1974] to refer to Green's function as a response function of the ocean.

When dealing with oceanographic problems, however, it is often difficult to construct the proper Green's function and transform methods are virtually indespensible in this case.

2-2- Green's Function for the Helmholtz Equation:

Before dealing with the Green's function for the Helmholz equation in a general form, we consider a simple example of this type the two dimensional Poisson equation

$$\nabla^2 \psi = \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = -4\pi\, \rho(X,y) \qquad (2.2.1)$$

inside the rectangular boundaries X = 0, X=a , y=0, y=b. First , we recapulate the eigenfunction method of solving the case for $\rho = 0$ (homogeneous equation) for homogeneous boundary conditions ($\psi = 0$) on the three side X=0, X=a, y=0 but with the inhomogeneous boundary conditions $\psi = \psi_b(X)$ along y =b The solution of (2.2.1) is:

$$\psi(X,y) = \int^{b} \psi_b(\xi)\, G^b\,(X,y/\xi)\quad d\xi \qquad (2.2.2)$$

Where the Green's function G^b $(X,y/\xi)$ may have the form

$$G^b(x,y\ /\xi) = \frac{2}{a}\sum \frac{\sinh\ (\pi ny/a)}{\sinh\ (\pi nb/a)} \sin\left(\frac{nx\pi}{a}\right) \sin\left(\frac{\pi n \xi}{a}\right) \qquad (2.2.3)$$

The quantity G^b , in the brackets is a function of the coordinates X and y and also of the position ξ on the surface y=b. It may be called the Green's function for boundary conditions on the surface y = b. Multiplying it by the specified boundary value for ψ and integrate over the boundary to obtain the solution. To study the Green's function for the Helmholtz operator

$$L\psi = \nabla^2\psi + k^2\psi = 0 \qquad (2.2.4)$$

for some bundary conditions on a closed surface S. We prove the following properties of Green's function:

1- The Green's function will be symmetric function of two set of coordinates, those of the observation point and those of the source point:

$$G_k(r/r_o) = G_k(r_o/r) \; ; \qquad \text{reciprocity relation} \qquad (2.2.5)$$

This function will satisfy some homogeneous boundary conditions on both S and S_o and will have a discontinivty at $r = r_o$.

2- Employing this function, it is possible to obtain a solution of the inhomogeneous equation with the given homogeneous boundary conditions or else the solution of the homogeneous equation with inhomogeneous boundary conditions.

Because of the linearity of the equation, we can also solve the inhomogeneous equation with inhomogeneous boundary conditions by superposition of both individual solutions.

3- The solutions for inhomogeneous boundary conditions will have a discontinuity at the boundary.

For instance, if ψ is specified on the surface (Dirichlet conditions), the solution will have the specified value ψ just inside the boundary and will be zero just outside. For Neumann conditions, where normal gradient is specified, the discontinuity in the gradient.

The required Green's function is the solution of the inhomogeneous Helmholtz equation

$$\nabla^2 G\,(r/r_o) + k^2\, G(r/r_o) = -4\pi\,\delta\,(r-r_o) \qquad (2.2.6)$$

for a unit point source at r_o, which satisfies homogeneous boundary conditions (either zero value or zero normal gradient of G) on the boundary surface S (and also in the source coordinates on S_o).

To show that the inhomogeneous equation

$$\nabla^2\psi + k^2\psi = -4\pi\rho(r) \qquad (2.2.7)$$

subject to arbitrary conditions on the closed boundary surface S may be expresed in terms of G, we multiply