ON THE APPROXIMATE SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION OF ORDER 2 BY AIRY'S INTEGRALS

THESIS

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بسم الله الرحمن الرحيم

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"[illegible], [illegible] AND MERCIFUL"

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NOTATIONS

1. $[m] =$ the integral part of m.

2. $(m)_r = m(m+1)(m+2) \ldots (m+r-1)$ (m,r are positive integers)

3. $(a,r) = a(a+1)(a+2) \ldots (a+r-1)$ (r positive integer)

   $(a,r) = (a)_r$ (a positive integer)

4. $Aix = \frac{1}{\pi} \int_0^\infty \cos(xt + t^3/3)\,dt$

5. $Bix = \frac{1}{\pi} \int_0^\infty [\sin(xt + t^3/3) + \exp(xt - t^3/3)]\,dt$

6. $u \equiv u(x) = c_1 Aix + c_2 Bix$ (c_1, c_2 arbitrary constants)

7. $u^{(n)} \equiv \frac{d^n}{dx^n} u(x) \equiv \frac{d^n}{dx^n} u$

8. $D \equiv \frac{d}{dx}$, $D^2 \equiv \frac{d^2}{dx^2}$

9. $\theta \equiv D^2 - x$

10. $\theta^{-1} \equiv$ inverse operator of θ

11.* x ^ 2 $\equiv x^2$, x ^ 3 $\equiv x^3$, x ^ 4 $\equiv x^4$

12.* y(n) $\equiv y_n$ the approximate solution of the nth order of the differential equation $y'' - (x+ax^m)y = 0$

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*The symbols appearing in (11) and (12) will be used in the last chapter "Computer Programmes".

INTRODUCTION

The integral $\int_0^\infty \cos(xt + t^3/3)\,dt$ appeared for the first time in the researches of Airy [1] on the intensity of light in the neighborhood of a caustic. This integral together with another one, discovered later, were shown to be definite integral solutions of the differential equation $y'' - xy = 0$. These definite integrals, denoted by Aix, Bix are

$$\text{Ai}x = \frac{1}{\pi} \int_0^\infty \cos(xt + t^3/3)\,dt,$$

$$\text{Bi}x = \frac{1}{\pi} \int_0^\infty [\sin(xt + t^3/3) + \exp(xt - t^3/3)]\,dt$$

and are called Airy's integrals.

In 1910, the famous British mathematician Hardy [2] obtained a generalization of Airy's integrals. Hardy has shown that if $\sinh z = s$, then

$$2\,{}^{\cosh}_{\sinh}\, nz = (2s)^n \times {}_2F_1\left(-\frac{n}{2}, \tfrac{1}{2} - \frac{n}{2}; 1-n, -\frac{1}{s^2}\right),$$

where

$${}_2F_1(a,b,c;x) = \sum_{r=0}^{\infty} \frac{(a,r)(b,r)}{r(c,r)}\, x^r ,$$

$$(a,r) = a(a+1)(a+2)\ \ldots\ (a+r-1),$$

and the cosh or sinh is taken according as n is even or odd. Now, if we write

$$T_n(t,\alpha) = t^n \times {}_2F_1(-\frac{n}{2}, \tfrac{1}{2} - \frac{n}{2}, 1-n; -\frac{4\alpha}{t^2}) ,$$

then

$$T_2(t,\alpha) = t^2 + 2,$$

$$T_3(t,\alpha) = t^3 + 3\alpha t, \ \ldots$$

$$\ldots \qquad \ldots$$

The following three integrals

$$Ci_n(\alpha) = \int_0^\infty \cos T_n(t,\alpha)\,dt,$$

$$Si_n(\alpha) = \int_0^\infty \sin T_n(t,\alpha)\,dt,$$

$$Ei_n(\alpha) = \int_0^\infty \exp[-T_n(t,\alpha)]\,dt,$$

are known as Airy-Hardy integrals. Among the various results obtained by Hardy, he has shown that if n (n > 2) is an odd integer $Ci_n(x)$ satisfies the differential equation

$$y'' - n^2 x^{n-2}\, y = 0$$

Accordingly for n = 3,

$$Ci_3(x) = \int_0^\infty \cos[T_3(t,x)]\,dt$$

$$= \int_0^\infty \cos(t^3 + 3xt)\,dt$$

satisfies the differential equation

$$y'' - 9xy = 0$$

which is also satisfied by $Ai(\sqrt[3]{9}\,x)$. In fact,

$$Ci_3(x) = \pi\, Ai(\sqrt[3]{9}\,x).$$

This makes clear the sense in which Hardy has generalized Airy's integrals.

In 1951, Yacoub [3], [4] generalized Airy's integrals but in a different sense. In fact, Yacoub obtained m definite integral solutions for the differential equation

$$y^{(m)} - xy = 0$$

When m is even, say m = 2n, it has been shown that among these 2n definite integral solutions there exist two definite integral solutions:

$$\frac{1}{\pi}\int_0^\infty \cos(xt + t^{2n+1}/(2n+1))\,dt,$$

$$\frac{1}{\pi}\int_0^\infty [\sin(xt + t^{2n+1}/(2n+1)) + \exp(xt - t^{2n+1}/(2n+1))]\,dt .$$

These two integrals reduce to Aix and Bix when n = 1. In this sense, the m definite integrals mentioned above represent generalization of Airy's integrals when m = 2n is even.

In the earliest fourtieth of this century, a question had been arisen by J.C.P. Miller if the reduced form of the second order linear differential equation namely

$$y'' - (x + a_2x^2 + a_3x^3 + \ldots)y = 0$$

may possess some sort of solution in terms of Airy's integrals Aix and Bix. An attempt towards the answer of this question has been done by Z. Mursi when the coefficients a_2, a_3, ... are small. The first three approximate solutions, denoted by

y_1, y_2, y_3, have been obtained by Mursi [5] in terms of Airy's integrals and their derivatives.

Moreover, when all the coefficients of powers of x greater than the second vanish, Mursi had obtained the first four approximate solutions y_1, y_2, y_3, y_4 for the simple differential equation

$$y'' - (x + ax^2)y = 0$$

For approximate solutions of higher order, even for the above simple equation, the situation seems complicated and laborious.

In 1967, Yacoub [6] obtained a process which simplifies the determination of approximate solutions of higher orders for the differential equation

$$y'' - (x + ax^m)y = 0$$

when $m = 2,3$.

In 1968, Yacoub [7] generalized these results when m is any integer greater than or equal to 2. In fact, a relation between the nth approximate solution y_n and the (n+1)th approximate solution y_{n+1} has been obtained.

It is the object of the present work to apply these results in the case $m = 4$. Moreover, three programmes have been used to obtain the approximate solutions:

$$y_1, y_2, \ldots, y_{20} \qquad \text{when} \quad m = 2,$$

$$y_1, y_2, \ldots, y_{18} \qquad \text{when} \quad m = 3,$$

and

$$y_1, y_2, \ldots, y_{12} \qquad \text{when} \quad m = 4$$

for the differential equation

$$y'' - (x + ax^m)y = 0.$$

The thesis contains five chapters.

Chapter I deals with Airy's integrals. It presents some miscellaneous results on Airy's integrals. Among these results the relation between Airy's integrals Aix, Bix (or their first derivatives) and Bessel functions have been introduced.

Chapter II is devoted to the expression of the higher derivatives $u^{(m)}$ of

$$u \equiv u(x) = c_1 \text{Ai}x + c_2 \text{Bi}x$$

in the form

$$u^{(m)} = P(x)u + Q(x)u^{(1)}$$

where $P(x)$ and $Q(x)$ are polynomials in x.

Formulae for the leading coefficients of these polynomials are given together with tables for such coefficients.

Chapter III deals with the second order linear differential equation in the reduced form (normal form)

$$y'' + f(x)y = 0$$

in general and in particular with the differential equation

$$y'' - (x + ax^m)y = 0$$

where a is small enough and m = 2,3.

In Chapter IV, we deal with the simple second order linear differential equation

$$y'' - (x + ax^m)y = 0$$

where m is any integer greater than or equal to 2 and a is small enough. In this chapter, the results of the previous chapter have been generalized. These results give, in fact, a relation between the nth approximate solution y_n and the (n+1)th approximate solution y_{n+1}.

In Chapter V, we give three computer programmes for the approximate solutions of the differential equation

$$y'' - (x + ax^m)y = 0$$

in the three cases: m = 2, m = 3 and m = 4.
These programmes enable us easily to obtain the successive approximate solutions in the three cases under consideration. For m = 2, we obtained the first 20 approximate solutions while for m = 3, we have obtained the first 18 approximate solutions, and for m = 4 we have obtained only the first 12 approximate solutions. This is due to the fact that for m = 3 or m = 4 the number of coefficients appearing in $y(n) \equiv y_n$ increases rapidly with n. For example when m = 3, y(18) contains 40 coefficients (Table 2) and when m = 4, y(12) contains 33 coefficients (Table 3).
But it should be noted that the above programmes apply as well to obtain approximate solutions of any higher order.

CHAPTER I

MISCELLANEOUS RESULTS ON AIRY'S INTEGRALS

CHAPTER I

MISCELLANEOUS RESULTS ON AIRY'S INTEGRALS

In this chapter, some well known miscellaneous results on Airy's integrals will be collected together. This chapter consists mainly of three sections. In the first section, we express Airy's integrals Aix and Bix as power series (see DeMorgan [8]). In the second section, we express the connection discovered by Stokes [9] between Airy's integrals and Bessel functions.

The third section is devoted to the particular integral of the differential equation

$$y'' - xy = F(x),$$

for certain forms of F(x). The results in this section are due to Mursi [5] (see also Yacoub and Mursi [10]).

SECTION I

AIRY'S INTEGRALS AS POWER SERIES

In this section, we describe two different methods to express the Airy's integrals Aix and Bix as power series. It should be observed that the second method used earlier by DeMorgan applies only for the Airy's integral Aix.

First Method

This method depends on the fact that Aix and Bix are both solutions of the differential equation.

(1) $$y'' - xy = 0$$

Now, to obtain the series solution of (1), let

$$y = \sum_{0}^{\infty} a_n x^{c+n}, \quad a_o \neq 0.$$

Proceeding in the usual way, we get:

(2) $$\sum_{0}^{\infty} (c+n)(c-n-1)a_n x^{c+n-2} - \sum_{0}^{\infty} a_n x^{c+n+1} = 0$$

Equating to zero the coefficient of x^{c-2}, we obtain

$$c(c-1)a_o = 0$$

which implies directly $c = 0$ or $c = 1$ (since $a_o \neq 0$). Equating to zero the coefficients of x^{c-1} and x^c we have

(3) $$(c + 1)c \quad a_1 = 0,$$

(4) $$(c + 2)(c + 1)a_2 = 0.$$

If we observe that $c = 0$ or 1, then (4) will give directly

(5) $$a_2 = 0.$$

Finally, if we equate to zero the coefficient of x^{c+n-2}, we get:

$$(c-n)(c-n-1)a_n - a_{n-3} = 0,$$

or equivalently

(6) $$a_n = a_{n-3}/(c+n)(c+n-1).$$