MATHEMATICAL MODELS FOR THE STUDY OF THE RELIABILITY OF SYSTEMS

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M.Sc. COURSES

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1- Theory of Probability and Designs
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4- Theory of Queues and Stochastic Processes
(2 hours per week)

PREFACE

The thesis deals with some mathematical models for the study of the reliability of systems. It consists of four chapters.

In chapter I we describe some basic laws and characteristics of reliability of systems. We also study the mathematical model of a two-unit warm-standby system and the limiting distribution of the residual lifetime. This chapter involves also a basic results showing that the conditioned limit distribution remains unchanged whatever be the initial state of the system and that it is always exponentially distributed. Kalpakam, S. and Shahul Hameed, M.A. (1983) have studied this problem. We have noticed some slight mistakes, and so we have studied this problem in details giving the corrections.

Chapter II is concerned with a renewable redundant system with arbitrary distribution for the volumes of the inputs. The transition probabilities of the states of the system are derived and an expression for the stationary distribution of the number of the units giving services, those in repair and those under maintenance are given. Moreover, the probability losses for the stationary state of the system is found. The material of this chapter is a generalization of the corresponding model discussed by S. Elias (1981).

Chapter III deals with mathematical models for calculating reliability functions and mean lifetimes for three models of standby redundant systems (loaded, non-loaded and lightly loaded, with repair and preventive maintenance). The results of G. Mokaddis and S. Elias (1978) have been obtained as special cases of our results. The material of this chapter has been accepted for reading in the Proceedings of the International Congress, the Institute of Statistical Studies and Research, Cairo University, 1984.

In chapter IV we study the reliability function and the mean lifetime for a three-unit standby redundant system with repair and preventive maintenance. Explicit expressions for Laplace transforms of the mean down time of the system during the period (0,t) and for the mean time to system failure have been obtained under the assumption that the repair and the preventive maintenance times of a unit are differently and arbitrary distributed. The results of S. Elias (1981), B.V. Gnedenko - Y.K. Belyaev - A.D. Solovyev (1969) and B.V. Gnedenko - M. Dwich - Y. Nasr (1975) are derived as special cases of the results of this chapter. Moreover, we compare between the mean lifetimes of the two-unit system and the three-unit system. The material of this chapter is completely new and has been published in the Proceedings of the Ninth International Congress of Statistics, Computer Science, Social and Demographic Research, Volume 7, Page 87 - 103, 1984.

CONTENTS

CHAPTER I

MATHEMATICAL MODEL OF A TWO-UNIT WARM-STANDBY RELIABILITY SYSTEM AND THE LIMITING DISTRIBUTION OF THE RESIDUAL LIFETIME

In this chapter, we introduce the basic laws and concepts of reliability of systems. We also deal with the residual lifetime distribution at time t, and the existence of its limit as $t \longrightarrow \infty$, of the two-unit warm-standby reliability system supported by a single repair facility, under the condition that the system has not been down at any instant in the interval $(0,t)$. The conditioned limit distribution has been obtained under the assumptions that all the distributions of the lifetimes and the repair times of the individual units are arbitrary, provided that their Laplace transforms are rational functions of their arguments. A basic result is obtained showing that the conditioned limit distribution remains unchanged whatever be the initial state of the system and that it is always exponentially distributed.

(1.1) SOME CHARACTERISTICS OF RELIABILITY OF SYSTEMS

Just as in other branches of science, the basic characteristics of reliability theory are understood by describing the relationships among them.

In this thesis, the word unit means an element, a system, a part of a system or the like. The reliability of a unit is connected in a very well way with the concept of its quality. The quality of a unit depends in a very well way on the manner in which it is used.

If τ is the lifetime of the operating unit, the unit begins to function at time $t = 0$ and a failure occurs at time $t = \tau$, τ being a non-negative random variable with probability density function $f(t)$ and zero for negative values of t, that is

$$f(t) = \lim_{t \to 0+} \frac{P(t < \tau < t + \Delta t)}{\Delta t} \qquad (1.1.1)$$

with

$$\int_0^{\infty} f(t)\, dt = 1 . \qquad (1.1.2)$$

The distribution of τ is determined by the probability density function, $f(t)$, but it is convenient to work with other functions equivalent to $f(t)$ such as $F(t)$ which gives the probability that a component has failed by time t. That is

$$F(t) = P(\tau \leqslant t) = \int_0^{t} f(u)\, du . \qquad (1.1.3)$$

$F(t)$ is known as the failure function. Equation (1.1.3) gives $F(t)$ in terms of the probability density function $f(t)$. Thus

$$f(t) = F'(t) = \frac{dF(t)}{dt} . \qquad (1.1.4)$$

It is more convenient to work with the function complementary to F(t). This is known as the reliability function or the survival function, R(t),

$$R(t) = P(\zeta > t)$$
$$= 1 - F(t)$$
$$= \int_t^{\infty} f(u)\, du \quad . \qquad (1.1.5)$$

R(t) gives the probability that a component has not failed up to time t. Clearly $R(0) = 1$, $R(\infty) = 0$ and R(t) is a non-increasing function of t. Also

$$f(t) = -R'(t) \quad . \qquad (1.1.6)$$

One of the important concepts in reliability theory is that of life. The mean lifetime T of a unit is defined by the mathematical expectation of the random variable

$$T = E(\zeta) = \int_0^{\infty} t\, f(t)\, dt$$
$$= \int_0^{\infty} t\, (-R'(t))\, dt$$
$$= \int_0^{\infty} R(t)\, dt \quad . \qquad (1.1.7)$$

So the mean lifetime of a unit is equal to the total area under the survival curve. The variance in the lifetime

$$\begin{aligned}
\mathrm{Var}(\tau) &= E(\tau - T)^2 \\
&= E(\tau^2) - T^2 \\
&= \int_0^{\infty} t^2 f(t)\, dt - T^2 \\
&= 2 \int_0^{\infty} t\, R(t)\, dt - T^2 \qquad (1.1.8)
\end{aligned}$$

describes the degree of dispersion in the value of the random variable τ about the mean lifetime T .

If a system fails with the failure of one of its components then it is called non-redundant system, otherwise the system is called a redundant system. By a standby redundant system we mean that the system is composed of main units and standby redundant units, and operating in a standby manner, that is, all the main units must operate to make the system in the operation state and if some of the main units fail they are replaced by units from the standby which