TAUBERIAN CONDITIONS INVOLVING TAUBERIAN THEOREMS

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CONTENTS

A B S T R A C T

The first part of this thesis contains a general account of the Tauberian conditions which have been defined by Schmidt, Pitt, Agnew and Soraya Sherif. It contains also, the simplified forms of Pitt's classes and the smaller classes than some of Agnew's which have been introduced by Soraya Sherif.

We discuss results of inclusion which hold for certain pairs of Tauberian classes.

Also, we prove that there is no relation at all between one of Pitt's and one of Agnew's classes. This has been proved by constructing an example. However, this example has been introduced by B. Kuttner in 1962.

But in this thesis we introduce a method of proof differs somewhat ~~of~~ than Kuttner's proof. Some further remarks on Tauberian conditions have been given to show whether a certain Tauberian condition holds for more than one Tauberian theorem or whether it holds for certain theorems and fails for others.

In the second part of the work, we show that one of Agnew's classes fails to establish two of Soraya Sherif's Tauberian theorems. We showed this by constructing an example. We also discuss Tauberian theorems introduced by Soraya Sherif, B. Kuttner, and Das.

The third part of this work, is concerned with the relation between the sequence-to-sequence and the series-to-series (S , μ_{n+1}) transformation.

Also, we introduce a new theorem for a transformation, of a more general kind. The theorem may be illustrated as follows :

Let

$$(1) \qquad t_n = \sum_{k=0}^{\infty} \alpha_{n,k} s_k \; ;$$

$$(2) \qquad t_n = \sum_{\nu=0}^{n} b_\nu \; , \qquad s_k = \sum_{\nu=0}^{k} a_\nu \; ,$$

and that

$\beta_{n,k} = \sum_{\nu=k}^{\infty} \alpha_{n,\nu}$. Suppose that $\beta_{n,k} s_k \longrightarrow 0$ as $k \longrightarrow \infty$

Then (1) is equivalent to the series-to-series transformation

$$(3) \qquad b_n = \sum_{k=0}^{\infty} \gamma_{n,k} a_k$$

where,

$$\gamma_{0,k} = \beta_{0,k} \; , \qquad \gamma_{n,k} = \beta_{n,k} - \beta_{n-1,k}$$

In the sense that if (1) converges for all n , then so does (3), and conversely, and that the sums are then related by (2).

Also, we will show in section two of chap. IV that, the result of K. Knopp and G.G. Lorentz [25] involving the equivalence of the sequence-to-sequence and the series-to-series Hausdorff transformation without any condition, is a special case of our new result mentioned above.

CHAPTER I

INTRODUCTION

1. Tauberian Conditions.

The sum of a series

(1.1) $$\sum a_n = a_0 + a_1 + \dots$$

is said to be convergent to the sum S if its partial sum

$S_n = a_0 + a_1 + \dots + a_n$ tends to a finite limit S when $n \longrightarrow \infty$, and a series which is not convergent is said to be divergent.

Thus the series

$1 - 2 + 4 - 8 + \dots$, $1 + 1 + 1 + \dots$

are divergent.

Newton and Leibniz, the first mathematicians to use infinite series had little temptation to use divergent series.

Afterwards, Euler has shown that the sum of the geometric series

$1 + x + x^2 + x^3 + \dots$ is $\frac{1}{1-x}$ if $|x| < 1$

Putting: $x = -1$ and $x = -2$,

James Bernoulli has also shown that :

$1 - 1 + 1 - 1 + \dots = \frac{1}{2}$

and $1 - 2 + 2^2 - 2^3 + \dots = \frac{1}{3}$ respectively.

Similarly from : $\left(\frac{1}{1-x}\right)^2 = 1 + 2x + 3x^2 + \dots$

he deduces the relation :

$$1 - 2 + 3 - 4 + \dots \qquad = \tfrac{1}{4} \,;$$

But the mathematicians of the 17th and 18th centuries had no clear insight into the reasons, why one type of result should be admitted, and not the other.

From this point of view, a general problem has been presented as follows :

A particular sequence (s_n) is defined in some way, either by direct indication of the terms, or by a series or product, or otherwise.

Is it possible to associate a "value" s with it, in a reasonable way ?

Let us illustrate this general statement by an example. The series

$$\sum (-1)^n = 1 - 1 + 1 - 1 + 1 \dots.$$

i.e. the geometric series $\sum x^n$ for $x = -1$, or the sequence

$$(s_n) \equiv 1, 0, 1, 0, 1, 0, 1, \dots$$

has so far been rejected as divergent, because its terms s_n do not approach a single definite number. On the contrary, they oscillate between 1 and 0.

This very fact, however, suggests the idea of forming the arithmetic means

$$(1.2) \qquad s'_n = \frac{s_0 + s_1 + s_2 \cdots s_n}{n+1} \qquad (n = 0, 1, 2, \dots)$$

since

$$s_n = \tfrac{1}{2}\left[1 + (-1)^n\right]$$, we find that

$$s'_n = \frac{(n+1) + \frac{1}{2}\left[1 + (-1)^n\right]}{2(n+1)} = \frac{1}{2} + \frac{1 + (-1)^n}{4(n+1)}$$

so that S'_n (in the former sense) approaches the value $\frac{1}{2}$, i.e. $\lim S'_n = \frac{1}{2}$.

This leads the mathematicians to introduce methods of summation or methods of limitations. Some of these processes are as follow :

Euler's process, or the E_1-process :

A sequence (S'_n) is said to be limitable E_1 with the value S, if the sequence (S'_n) tends to S where

$$(1.3) \qquad S'_n = \frac{1}{2^n}\left[\binom{n}{0}S_0 + \binom{n}{1}S_1 + \qquad + \binom{n}{n}S_n\right]$$

Cesaro's process, or the C_k-process

We first write $S_n = S_n^{(0)}$, and also, for each $k \geq 0$

$$S_0^{(k-1)} + S_1^{(k-1)} + \dots \qquad + S_n^{(k-1)} = S_n^{(k)}, \qquad (n = 0,1,2,\dots)$$

and we now examine the sequence of numbers

$$(1.4) \qquad C_n^{(k)} = \frac{S_n^{(k)}}{\binom{n-k}{k}},$$

for each fixed k. If, for some value of k, $C_n^{(k)} \longrightarrow S$ we say that the sequence (S_n) is limitable C_k with the value S.

Borel's process, or the B-process.

Considering the two series

$$\sum_{n=0}^{\infty} S_n \frac{x^n}{n!} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

that is to say, the product.

$$(1.5) \qquad F(x) = e^{-x} \cdot \sum_{n=0}^{\infty} S_n \frac{x^n}{n!}$$

for $x \longrightarrow +\infty$, we obtain the process introduced by

E. Borel. In accordance with it we make the following definition :

A sequence (S_n) such that the power series $\sum S_n \frac{x^n}{n!}$ converges everywhere and the function $F(x)$ just defined tends to a unique limit S as $x \longrightarrow +\infty$, will be called limitable B with the value S .

Abel's process, or the A-Process.

Given a series $\sum a_n$ with the partial sums S_n , we consider the power series

(1.6) $$F(x) = \sum a_n x^n = (1-x) \sum S_n x^n$$

[see Knopp page [5] chapter (V)]

If its radius is $\geq$ 1, and if (for real values of x) the limit

(1.7) $$\lim_{x \longrightarrow 1-0} \sum a_n x^n = \lim_{x \longrightarrow 1-0} (1-x) \sum S_n x^n = S$$

exists, we say that the series $\sum a_n$ is summable A , and that the sequence (S_n) is limitable A , with the value S; in symbols :

(1.8) $$A - \sum a_n = S \, ; \qquad A - \lim S_n = S \, .$$

General transformation

A linear transformation

$$t_m = \sum K_{m,n} s_n \tag{1.9}$$

of the sequence s_n into the sequence t_m is called Abelian if, whenever s_n converges, t_m converges to the same limit.

In general, the convergence of t_m will not imply the convergence of s_n, but it is frequently possible to state a supplementary condition on s_n, which, together with the convergence of t_m, implies the convergence of s_n. Such a supplementary condition is usually called a Tauberian condition on s_n, at least if it takes the form of some restriction on the rapidity of oscillation of s_n.

Similar considerations apply to transformations

$$t(x) = \int K(x,y)\, s(y)\, dy \tag{1.10}$$ +

of a function $s(x)$ into a function $t(x)$.

The conditions gain their name from a theorem of A. Tauber [48], to the effect that if (1.7) holds, and if

$$\alpha_n = o\left(\frac{1}{n}\right) \tag{1.11}$$

then

$$\sum_{0}^{\infty} \alpha_n = s \tag{1.12}$$

This is a conditioned converse of Abel's theorem,

+ Here, and throughout the thesis, integrals without limits are over $(-\infty, \infty)$.

which states that (1.7) follows from (1.12), without the mediation of any hypothesis as (1.11).

Later, Littlewood [28] proved that in the above theorem of Tauber the condition (1.11) can be replaced by the broader condition

$$a_n = O\left(\frac{1}{n}\right)$$

It was afterwards shown by Hardy and Littlewood [12] that it is sufficient to suppose that

$$n a_n > -K$$

The hypothesis of the theorem was generalized in a somewhat different direction by Landau [27], who proved that the conditions

$$s_n = O(1),$$

$$\lim_{\delta \to 0} \ \underset{n(1+\delta) \leq m \leq n(1+\delta)}{\text{maximum}} |s_m - s_n| = 0$$

are sufficient.

Later, R. Schmidt [42] proved a theorem which includes all these generalizations, in which it is supposed only that

$$\underline{\lim} \ (s_m - s_n) \geq 0$$

when $m > n$, and m and n tend to infinity so that $m/n \to 1$.

But in (1938) H.R. Pitt [34] introduced Tauberian classes of a more general kind than that of Schmidt.

In (1941) R.P. Agnew [1] introduced also Tauberian classes.

In (1964) S. Sherif [45] has established the relation between Agnew, and Pitt's classes. Also, she introduced

Tauberian classes related to those of Pitt [34] and Agnew [1].

We will be concerned in section one of chap. II, with the Tauberian classes of Schmidt [42], Pitt ([34], [35], [36]), Agnew [1] and Sherif [45].

We shall also discuss the simple forms which Sherif [43] gave to the definitions of the classes T, T_α and T_β of Pitt [36], which will be defined by (1.10), (1.13) and (1.14) respectively in chap. II.

Also, we shall give the definitions of the classes $A(\theta)$, $A^*(\Delta)$ of Agnew which has been introduced by Sherif [43] and which will be defined by (1.15) and (1.22) of chap. II of this thesis respectively.

These definitions of Sherif's are less general than the classes T, T^* of Agnew [1]. A definition to the class $A'_\rho\,(0 < \rho \leqslant 1)$ of Agnew will be given by (1.23).

Agnew proved that $A \subset A^*$, but Sherif [44] has shown that, $A(\theta) \subset A^*(\frac{\pi}{2}-\theta)$, so that Agnew's result will be as corollary of her's. We introduce also the Tauberian classes C, C', C'', C_1, C'_1 and C''_1 of Sherif [45]. We may note that until now no Tauberian classes wider than those of Sherif's have been introduced by any one in the field of Tauberian conditions.

In section two of chap. II, we consider the relation between Pitt's, Schmidt's, Agnew's and Sherif's classes. Theorem 2.1 involves the relation between Pitt's and Schmidt's classes which has been shown in Pitt [36]. But Theorem 2.2 involves the relation between the $T_\alpha\,(\mu=0)$ of Pitt [36] and $A^*(\frac{\pi}{2})$ of Agnew.

Then

$$\lim_{y \to \infty} s(y) = A$$

The main theorems of [34] are generalizations of Theorem "a". For example, he showed that all the familiar Tauberian conditions, including the "high-indices" conditions, can be regarded as special cases of the more general condition defining his class T, and that theorem "a" remains true with this condition instead of (v).

In (1948) Pitt [35] has given three elementary Tauberian theorems for the transformation.

(2.1) $$g(u) = \int k(u,y)\, s(y)\, dy,$$

where, except in the third theorem, the kernel $k(u,y)$ satisfies the conditions

(2.2) $$k(u,y) \geq 0,$$

(2.3) $$\int k(u,y)\, dy = 1 \qquad \text{for all real } u,$$

(2.4) $$\lim_{u \to \infty} \int_{-\infty}^{Y} k(u,y) dy = 0 \qquad \text{for every fixed } Y.$$

The first theorem shows that if $s(x)$ is bounded and belongs to T, and if $\Lambda(\epsilon) > \frac{1}{2}$, where

$$\Lambda(\epsilon) = \lim_{v \to \infty} \underline{\lim}_{x \to \infty} \; \underset{u > v}{U \cdot b} \int_{\xi-\delta}^{\xi+\epsilon} k(u,y)\, dy$$

Then if

$$S = \overline{\lim_{x \to \infty}} |s(x)|; \qquad G = \lim_{x \to \infty} |g(x)|,$$

we have

(2.5) $$S\,[2\Lambda(\epsilon) - 1] \leq G + \epsilon$$

and, if $\Lambda(+0) > \frac{1}{2}$

$$s \leqslant \frac{G}{2\Lambda(+0) - 1} \tag{2.6}$$

The second theorem shows that if the same hypotheses hold, $s(x)$ belongs to T_∞ and that $\Lambda(\epsilon) = 1$ for every positive ϵ, then the core* of $s(x)$ (the least closed convex set containing the set of limit points of $s(x)$) coincides with the core of $\sigma(x)$.

The third and final theorem replaces the hypothesis that $K(u,y) \geqslant 0$ by the hypothesis that

$$\int | K(u,y) | \, dy \leqslant M, \qquad -\infty < u < \infty$$

(2.3), (2.4) hold, $s(x)$ is bounded and belongs to T_β , and that

$$\lim_{v \to \infty} \lim_{x \to \infty} \text{L.b.}_{u > v} \left\{ \int_{-\infty}^{\xi-\delta} + \int_{\xi+\delta}^{\infty} \right\} | K(u,y) | \, dy = 0 \tag{2.7}$$

Then every limit point of $s(x)$ is a limit point of $\sigma(x)$.

A remark in the introduction of Pitt [35] must be amended. It is said that if $K(x)$ is integrable over $-\infty < x < \infty$, if $s(x)$ is bounded and differentiable and such that $s'(y) \to 0$ as $y \to \infty$, and if

$$\lim_{u \to \infty} \int K(u-y) \, s(y) \, dy \tag{2.8}$$

exists, then $\lim_{y \to \infty} s(y)$ exists "whatever the form of $K(x)$".

It must be assumed that $\int K(x)\,dx \neq 0$ because if the integral is "0" then the limit in (2.8) is zero for all

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* (Cooke, R.G. [5] Chap. 6)

similary hypotheses may bear on ... on ... or on