

Ain Shams University



IN THE NAME OF GOD , THE MOST COMPASSIONATE , THE MOST MERCIFUL

ASPECTS OF PURITY AND PURE QUASI-INJECTIVITY IN THE CATEGORY OF MODULES

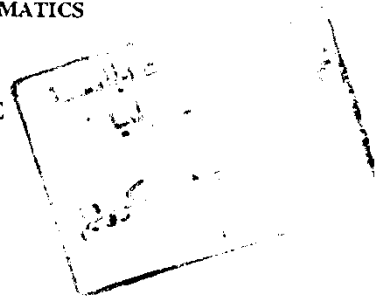
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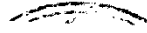
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INTRODUCTION

INTRODUCTION

In this thesis certain aspects of purity of some classes of algebraic structures are investigated in considerable detail. The results presented and established in the three chapters that comprise this work are - up to our knowledge and apart from some minor known results cited in the appropriate places in the dissertation - all new. The reader is kindly referred to [12] , [13] , for basic notions and results used here without explanation. Symbols and abbreviations used throughout are listed in the last page of the dissertation.

Pure subgroups and pure submodules play important and vital roles respectively in Abelian Group Theory and Module Theory. This can be seen if one effects localization at a prime integer or a prime ideal ; and in this context we can get, for example, a sharp classification of certain classes of torsion-free abelian groups. Another motive that initiated and introduced such study is the existence of some sort of affinity between the celebrated Baer's injectivity criterion and what we call pure Baer injectivity. Furthermore we felt that some sort of a general radical theory for purity can be initiated. Though chapter III , in which we take a step toward this , study does not cover all aspects of purity , we are inclined to think that such a study is worthwhile to be considered.

In this introduction we sum up the most important results we have obtained. The interested reader will find further numerous results embodied in this work.

Throughout , all rings are understood to be associative rings with identity elements ; and over such rings modules are , except at cited places , left modules and in all cases are unitary modules.

In the first chapter we generalize the notion of pure injectivity of modules by introducing what we call a pure Baer injective module. Theorem 1.10 gives a sufficient and necessary condition so that every left R -module should be a pure Baer injective R -module. Direct sums and direct products of such modules are dealt with in propositions 1.3 and 1.4. Theorem 1.6 singles out a homomorphically closed class of pure Baer injective R -modules. We also introduce two notions closely related to pure Baer injectivity ; namely the notions of a Σ -pure Baer injective module and that of SSBI-ring. A ring R is an SSBI-ring if and only if every semisimple R -module is pure Baer injective. To investigate such algebraic structure we were bound to define what we call , P -essential extensions , pure relative complement submodules and left pure hereditary rings. The basic properties of these concepts are given , together with their combining relations. For example theorem 1.23 links left pure hereditary rings with left pure-split rings ; whereas the counter-example exhibited in remark 1.13 shows that pure-split rings differ from semisimple rings in the sense that they cannot be characterized completely in terms of the class of pure-split modules. We devote as well a section , namely section two, for what we call p -essential submodules ; a notion that will be seen necessary and important in both combining and introducing related results. On the other hand theorem 2.17 characterizes commutative V -rings in terms of p -essential ideals of the concerned rings. For a characterization of pure injective modules we refer the reader to proposition 2.8. Proposition 2.15 gives a conclusive condition for a commutative ring to be semisimple. Finally proposition 2.14 shows that every submodule of a given module possesses a pure relative complement in that module.

Chapter two is divided into three sections namely sections 3, 4 and 5. In section three the notion of a pure Baer projective module is introduced as a dual concept of that of a pure Baer injective module. Theorem 3.3 gives a criterion for the class of a pure Baer projective

modules to be hereditary. In analogy with related classical results and with apparent different proof theorem 3.7 shows that if every module over a given ring is pure Baer injective then every module over such a ring is pure Baer projective and vise-versa.

Section four deals with what we call left FPI-rings , namely those rings over which every flat left module is pure Baer injective. We prove that left FPI-rings satisfy the ascending chain condition (A.C.C.) on pure left ideals (theorem 4.2). Theorems 4.3 , 4.4 , 4.5 , 4.6 and 4.7 brings , up to isomorphism , a close relationship between both left FPI-rings and left pure-split rings , perfect rings and left semisimple rings. In corollary 4.6 we show that a Von-Neumann regular left FPI-ring is Jacobson semisimple. We could not completely characterize the homomorphically class of the class of left FPI-rings ; however theorem 4.9 gives a partial answer of this question.

Section five is devoted to the two notions of pure quasi-injective modules and that of Π -pure quasi-injective modules. Propositions 5.2 and 5.9 encompass the classes of injective , pure Baer injective and pure quasi-injective modules within a common language under slightly imposed conditions. A fully invariant pure submodule of a pure injective module can be identified via pure quasi-injectivity see proposition 5.3 . Lemma 5.5 shows that a direct summand of a pure quasi-injective module is again pure quasi-injective.

The last chapter , namely chapter three which contains only one section , introduces and investigates what we call the pure radical $P(M)$ of a given module M . The pure radical $P(M)$ is the intersection of all maximal pure submodules of M . In theorem 6.14 the pure radical of M is interpreted in terms of the kernel of all the homomorphisms of onto pure simple modules which has a pure kernel. In theorem 6.3 we prove that $P(M/P(M)) = \{0\}$. More generally theorem 6.4 gives us a

necessary and sufficient condition under which the pure radical of a factor module of a module with respect to a pure submodule should be zero. The importance of this result can be visualized in view of the example given in the section in relation with theorem 6.4. Pure small submodules of a module are introduced to shed more light on the theory (see results 6.7, 6.8 , 6.9 and 6.10). Some modules that have zero pure radicals are given by 6.22. Theorems 6.15 , 6.17 , 6.19 and 6.20 consider specific homomorphisms $\phi : M \rightarrow N$ and show that $P(N) \supseteq \phi(P(M))$ in each of these cases ; equality holds in some cases. The chapter contains further results concerning pure radicals and we kindly refer the reader to section five for the details.

Finally we would like to assert that we confined ourselves by presenting only the results we have already obtained; bearing in mind that the introductory chapter-namely chapter 0 - gives all the necessary ingredients.

CHAPTER 0

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

2. In the second part, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[0, 1]$. It is shown that the maximum value is attained at $x = 0$ and is equal to 1.

3. The third part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

4. In the fourth part, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[0, 1]$. It is shown that the maximum value is attained at $x = 0$ and is equal to 1.

5. The fifth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

6. In the sixth part, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[0, 1]$. It is shown that the maximum value is attained at $x = 0$ and is equal to 1.

7. The seventh part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

8. In the eighth part, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[0, 1]$. It is shown that the maximum value is attained at $x = 0$ and is equal to 1.

9. The ninth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

10. In the tenth part, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[0, 1]$. It is shown that the maximum value is attained at $x = 0$ and is equal to 1.