NUMERICAL TREATMENT OF SYSTEMS OF LINEAR EQUATIONS

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THESIS

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By

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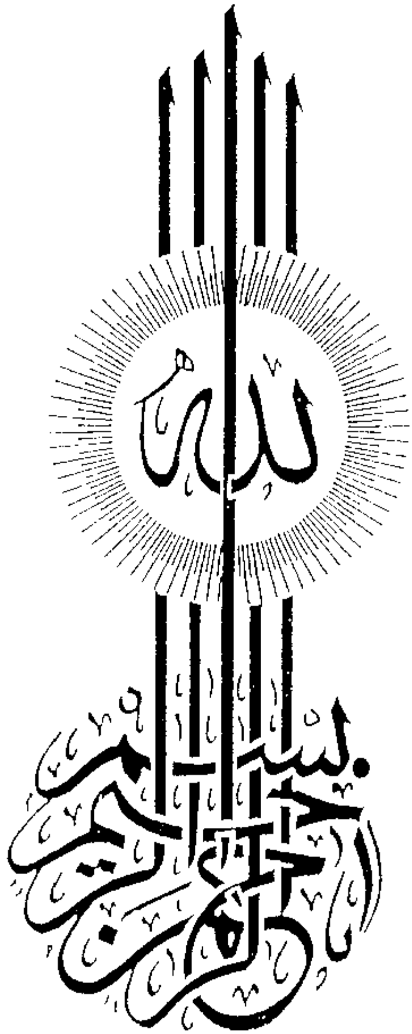
TO MY HUSBAND


بسم الله الرحمن الرحيم

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SUMMARY

This thesis contains five main chapters. The first chapter is considered as fundamentals for solving sets of linear simultaneous equations. The most important problem of numerical linear algebra is the development of algorithms for the solution. Essentially there are two reasons why this is such a basic problem. First of all, today nearly every linear problem in applied mathematics – as a boundary value problem for a linear ordinary or partial differential equation is reduced by appropriate techniques to a system of linear equations, especially when electronic computers are to be employed for its solution. Secondly, nonlinear problems can in most cases be solved by approximating them by linear problems.

There are several different methods for the solution of sets of equations, the method used depending on the calculating aids available, the type of equations to be solved and the accuracy required in the solution. In general, there are two types of numerical techniques for solving simulataneous linear equations: direct methods, which are finite, and indirect methods which are infinite. The two methods are useful, both have advantages and limitations.

In the second chapter, the Gaussian elimination method is treated practically in the matrix formulation. The accuracy of the solution depends on the accuracy with which complet working has been carried out. In practice, because a computer works with a finite word length, this method does not lead to exact solutions. Indeed errors arising from roundoff, instability, and loss of significance may lead to poor or even useless results. A large part of numerical analysis is concerned with why and how these

errors arise, and with the search for methods which minimize the totality of such errors. A method of refining the solution is derived in which the real interest lies in its application to a solution obtained with row interchanges, that nevertheless contains accumulated roundoff error. The effect of errors in the coefficients on the solution of the linear equations is discussed. The Gaussian elimination is represented by a flow chart and a corresponding computer program.

We studied the ill conditioned system, where small changes is the values of the coefficients cause very large changes in the solution. Several measures for ill conditioned equations are proposed and a method for solving such equations is discussed.

The third chapter deals with a more common technique which is known as the Gauss-Seidel iteration method. This technique is developed in the general form using the most recently computed values of the solution. It is extremely well adapted for use on a digital computer. This method would in principle require an infinite number of arithmetic operations to produce an exact solution. Thus, the accuracy of the solution depends on the number of iterations. The important advantages of this method are the simplicity and uniformity of the operations to be performed, which make them well suited for use on computers, and their relative insensitivity to the growth of round off errors.

The convergence analysis of Gauss-Seidel method is studied. A modified method is derived to speed up the convergence of the algorithm. Moreover, some divergent cases are turned into convergent ones. Many systems of equations arising in practice are sparse and iteration-if it works - is highly preferable. The Gauss-Seidel method is represented by a flow chart and is translated by a corresponding subroutine computer program.

The fourth chapter is concerned with the bandtype systems which arise frequently in solving ordinary differential equations and boundary value problems by difference methods. The partial differential equations can also be approximated to a system of linear algebraic equations in the band form. Practical methods for solving large tridiagonal and five-diagonal systems of linear equations are derived. These methods are represented by flow charts and their corresponding computer programs to be used directly.

In the fifth chapter, the Gauss-Jordan complete elimination method is expressed in a compact matrix form. A procedure is derived to alleviate serious errors introduced when any of the diagonal elements is small in magnitude compared with the other elements. This method is represented by a flow chart. A modified procedure is developed to save a portion of computer storage and is represented by a flow chart.

When several systems are to be solved, all of which have the same coefficient matrix but different right-hand side column constant vectors, then it is most valuable to obtain the solution by matrix inversion and multiplication. It is necessary to calculate the inverse only once in this case through the use of an appended unit matrix and to use it as a premultiplier of each of the right hand-side column vectors. This procedure is represented by a compact flow chart. A practical problem, voltages and currents in an electrical network is solved numerically for different sets of data.

CHAPTER I

Fundamentals

The interest in numerical analysis has increased tremendously in recent years, primarily because of the success of the large-scale calculating machines. The ability to obtain numerical answers in a short time has led engineers and scientists into lengthly analytical studies, which in turn have placed additional emphasis on numerical methods.

The most important problem of numerical linear algebra is the development of algorithms, that is, of arithmetical procedures - for the solution of systems of linear equations with many unknowns. Essentially there are two reasons why this is such a basic problem. First of all, today nearly every linear problem in applied mathematics-as a boundary-value problem for a linear ordinary or partial differential-equation-is reduced by appropriate techniques to a system of linear equations, especially when electronic computers are to be employed for its solution. Secondly, nonlinear problems can in most cases be solved by approximating them by linear problems.

In a set of simultaneous linear equations, frequently, the number of equations will be equal to the number of variables, in such case, as is to be expected, we are usually able to solve for unique values of the variables. If there are more variables than equations, we expect, in general, to obtain a finite number of solutions. Sometimes, we have more equations than variables. Thus for the d-c circuit, it is possible to write down many more equations than there are variables. However, not all of these equations are independent since some of them can be obtained from the others. Under such circumstances, it is desirable to find enough independent equations to be able to solve for all the variables.

1.2. Solvability of systems of linear equations

Let us consider a set of m simultaneous linear equations in n unknowns in the form

$$Ax = b$$

where A is an $m \times n$ matrix, $x = (x_j)$ and $b = (b_i)$, $j = 1(1)n$ and $i = 1(1)m$.

We are concerned with deriving criteria for the existence and uniqueness of solutions and with the properties of solutions.

We cannot say without investigation that there is a solution or, if there is one, that is unique. There are three and only three possibilities.

1. The system has a unique solution
2. The system has no solution
3. The system has an infinite number of solutions

A system of type 2 or 3 is said to be singular. Sometimes we know from the formulation of a problem that the system cannot be singular. If this formulation is lacking, as it often is, we must either depend on the method of solution to signal singularity or make an explicit test for the possibility.

(i) If we have n equations in n unknowns, i.e. m=n, we shall see later that Gaussian elimination does give immediate information about singularity. A direct test is to compute the coefficient determinant of the system, a zero value indicates singularity. Unfortunately, evaluating the determinant is as much work as solving the system.

The case in which det $(A) = 0$ is evident through the fact that after r such eliminations, where r is the rank*) of the coefficient matrix, all coefficients in the n-r succeeding equations will vanish (except for round off). Unless all right-hand members of those equations are also reduced to zero at that stage, the original set will be unsolvable. If all those members are zero, the n-r equations are reduced to $0 = 0$, and can be ignored. The rth equation would express x_r as the sum of a specified constant and a certain linear combination of $x_{r+1}, \dots, x_n$, and the process of backward substitution finally expresses $x_1, x_2, \dots, x_r$ in similar form.

(ii) In the general case, where no restriction will be made as to whether $m > n$, $m = n$, $m < n$, we

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*) The rank of a matrix is defined as the largest order of the nonvanishing determinant that can be formed from the matrix.