THE PROPAGATION OF ALFVEN'S WAVES IN THE EARTH'S CORE (ONE OF THE UNSYMMETRIC CASES)

A THESIS

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By

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بسم الله الرحمن الرحيم

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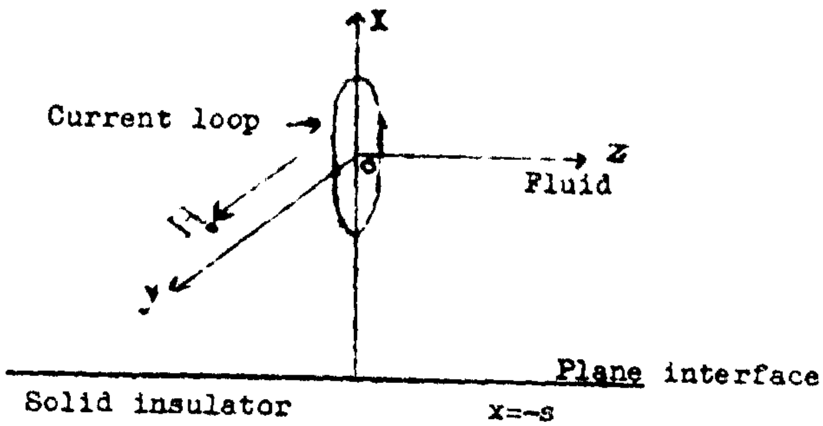
such that:

$$\underline{M} = \lim_{a \to 0} \pi a^2 \underline{I}$$

$$\underline{I} = \underline{i}_\phi \, I \, \delta(\rho - a) \, H(t)$$

where $\underline{i}_\phi$ is a unit vector in the direction of ϕ increasing, (ϕ, ρ, Z) are cylindrical polar coordinates; H(t) is the Heaviside unit function



The direction of the exitation field H_o is taken in the direction of the y-axis parallel to the plane boundary of the insulator perpendicular to the dipole axis.

Ma the matical solutions for the arising magnetohydrodynamic disturbances at the plane interface between the insulator and the fluid are given as well as in the insulator itself.

This work is a continuation to the work previously done by Mrs. F. Metwally & Mrs. S.N.Boctor where

I. INTRODUCTION

It is supposed that initially we have at rest a semi-infinite, non-viscous, incompressible fluid of finite electrical conductivity σ , density ρ and hydrostatic pressurep with a uniform field H_o prevailing through it.

A magnetic dipole is supposed to be suddenly introduced in the fluid at zero time to act as a source of distrubance to the original configuration of the system. By doing so, a small velocity $\underline{U}$ is produced in a certain volume of the fluid and the magnetic field becomes

$$\underline{H} = \underline{H}_o + \underline{h} \qquad (1)$$

where $\underline{h} \ll \underline{H}_o$

and $\quad \text{div } \underline{h} = 0 \qquad (2)$

The dipole is considered to be fixed at the origin of coordinates with its axis always directed along the z-axis.

The dipole can thus be replaced by a small fluid current loop of strength $\underline{I}$, radius a, placed in the (x,y)-plan with its centre at the arigin of coordinates.

The moment of the dipole is

$$\underline{M} = \lim_{a \to 0} \pi a^2 \underline{I} \qquad (3)$$

The electric current intensity of the loop is

$$\underline{I} = \underline{I}_\phi \; I \; (\rho - a) \; H(t) \qquad (4)$$

Where $\underline{I}_\phi$ is a unit vector in the direction of ϕ increasing; (ρ, ϕ, Z) are cylindrical polar coordinates; and $H(t)$ is the Heaviside unit function.

The fluid is bounded by an insulator with its rigid interface taken in our case as the plane $X = -s$, while the exitation field $\underline{H}_o = H_o \underline{I}_y$ is considered to be perpendicular to the dipole axis but parallel to the plane boundary. (see figure in Summary)

Mathematical representation for the small disturbances , $\underline{h}$, at the plane interface between the insulator and the fluid are given as well as in the insulator itself.

II. METHOD OF ANALYSIS

§ 2.1. Basic Equations

The phenomena can be described by using Maxwell's equations and the Eulerian hydrodynamic equation.

Maxwell's equations in the presence of a moving electrically conducting matter (e.m.v. being used) are

$$\text{Curl } \underline{E} = - \partial \underline{H} / \partial t \qquad (5)$$

$$\text{Curl } \underline{H} = 4\pi \underline{J} \qquad (6)$$

$$\underline{J} = \sigma (\underline{E} + \underline{U} \wedge \underline{H}) \qquad (7)$$

where the electric field $\underline{E}$, the electric current $\underline{J}$, the magnetic field $\underline{H}$, and the fluid velocity $\underline{U}$ are each a solenoidal vectors.

The Eulerian hydrodynamic equation for an incompressible, non-viscous fluid is

$$\frac{d\underline{U}}{dt} = \underline{F} - \frac{1}{\rho} \text{ grad } p \qquad (8)$$

where $\underline{F}$ is the external force per-unit volume.

To express our results in terms of non-dimensional units, we measure the magnetic field in terms of $\underline{H}_0$, the fluid velocity U in units of V, time in units of $\frac{1}{w_o}$, the hydrostatic pressure p in units of ρV^2 and length in units of V/w_o where V is the Alfvén velocity given by:

$$V^2 = H_o^2 / 4 \pi \rho \tag{9}$$

and w_o is a frequency defined by

$$w_o = \sigma H_o^2 / \rho \tag{10}$$

Making use of expression (1), rembering that $\underline{H}_o = H_o \underline{I}_y$ equations (5-7) showthat

$$\text{Curl } \underline{E} = - \partial \underline{h} / \partial t \tag{11}$$

$$\text{Curl } \underline{h} = \underline{E} + \underline{U} \wedge \underline{I}_y \tag{12}$$

Taking the curl of equation (12) substituting from equation (11), we see that

$$\frac{\partial \underline{h}}{\partial t} - \nabla^2 \underline{h} = \text{Curl } (\underline{U} \wedge \underline{I}_y) \tag{13}$$

Also equation (8) shows that

$$\frac{\partial \underline{U}}{\partial t} = \frac{\partial \underline{h}}{\partial y} + \underline{P} \tag{14}$$

where, except on the loop itself

$$\underline{P} = - \nabla \overset{\vee}{w} \tag{15}$$

and

$$\overset{\vee}{w} = p + \frac{1}{2} (\underline{I}_y + \underline{h})^2$$

Thus

$$\text{Curl } \underline{P} = 0 \tag{16}$$

Equations (13) and (14) are the fundamental equations of the problem.

It follows that solenoidal vectors $\underline{\Psi}$, $\underline{\psi}$ and $\underline{\zeta}$ should exist such that

$$\underline{h} = \text{curl } \underline{\Psi} \;, \; \underline{U} = \text{curl } \underline{\psi} \; \text{ and } \; \underline{P} = \text{curl } \underline{\zeta} \tag{17}$$

and

$$\underline{U} \wedge \underline{I}_y = \frac{\partial \underline{\psi}}{\partial y} - \nabla \underline{\psi}_y \tag{18}$$

Also a scalar potential of E exists such that

$$E = -\partial \Psi / \partial t + \nabla \alpha \tag{19}$$

where

$$\nabla^2 \alpha = 0 \tag{20}$$

Equation (13) now shows, on making use of relation (17-19), that

$$(\nabla^2 - \partial/\partial t) \Psi = -\partial \psi / \partial y + \nabla \psi_y - \nabla \alpha \tag{21}$$

Equation (14) also gives rise to

$$\partial \Psi / \partial y = \partial \psi / \partial t - \zeta - \nabla B \tag{22}$$

where

$$\nabla^2 B = 0 \tag{23}$$

The fundamental equations (13) and (14) are now expressed in the form given by equations (21) and (22) which describe the hydromagnetic disturbances and their effect on fluid motion.

Also since no current can flow in an insulator, it follows that

$$\text{Curl } \underline{h} = 0 \qquad (24)$$

Thus the disturbed field, h, can be represnted in the insulator as the gradient of a magnetic scalar potential, Ω , say; and as h is solenoidal, it follows that

$$\nabla^2 \Omega = 0 \qquad (25)$$

§ 2.2. Boundary Conditions

As the equations are linear, we can break up solutions of the problem into two parts: That which includes singularity at the loop as if it were placed in an infinite fluid and that which contributes to the presence of the insulator with its plane rigid interface bounding the fluid.

A - Boundary conditions on Crossing the Loop

It is to be noted that the only body force in our case is $\underline{J} \wedge \underline{H}$ which is the mechanical force exerted by the magnetic field $\underline{H}$ on a volume element of the fluid carrying a current $\underline{J}$. Thus the Ponderomotive force given by

$$\underline{F} = \underline{J} \wedge \underline{H} - \underline{I} \wedge \underline{H} \qquad (26)$$

where we have added a force $- \underline{I} \wedge \underline{H}$ to balance the Lorentz force on the loop, thus preventing its motion. Also if $\underline{\xi}$ be the electromotive force of the current loop, then the effect of the disturbance can be examined by introducing $\underline{\xi}$ as an extra term in equation (7) and by substituting for F as given in equation (26) in the hydrodynamic equation (8).

By doing so, the singularities at the loop have now been introduced in the fundamental equations (13) and (14).

Thus we have

$$\frac{\partial \underline{h}}{\partial t} - \nabla^2 \underline{h} = \frac{\partial \underline{U}}{\partial y} + \text{Curl } \underline{I} \qquad (27)$$

$$\frac{\partial \underline{U}}{\partial t} = \frac{\partial \underline{h}}{\partial y} - \underline{I} \wedge \underline{I}_y + \underline{P} \qquad (28)$$

which on using relations (17), show that on introducing the singularities at the loop, equations (21) and (22) lead respectively to

$$\left(\frac{\partial}{\partial t} - \nabla^2\right) \underline{\psi} = \frac{\partial \underline{\chi}}{\partial y} + \underline{I} - \nabla(\chi_y - \alpha) \qquad (29)$$

(iv) $\partial h_n/\partial n$ is continuous from (4.5) and the equations of continuity of $\underline{h}$.

§ 2.3. Integral Transforms Used;

1. The Heaviside transform denoted by a bar over the transformed symbol. It is defined by

$$\bar{a}(P) = p\int_0^\infty e^{-pt}\, a(t)\, dt \qquad (33)$$

Integration by parts shows that

$$\partial \underline{a}/\partial t = p\,\bar{\underline{a}} \;,\quad \partial^2 \underline{a}/\partial t^2 = p^2\,\bar{\underline{a}} \qquad (34)$$

Since initially we have the field to be uniform and the fluid is at rest.

2. The double Fourier transform denoted by a superscript star, thus:

$$a^*(\ell, m, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(x,y,z)\, \exp\, i(\ell x+my)\, dx\, dy \qquad (35)$$

the solution to which is

$$a(x,y,z) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} a^*(\ell,m,z)\exp -i(\ell x+my)\, d\ell\, dm \qquad (36)$$

3. The Fourier transform denoted by a superscript dagger, thus:

$$a^{+}(x,y,n) = \int_{-\infty}^{\infty} a(x,y,z)\ \exp(inz)\ dz \qquad (37)$$

the solution to which is

$$a^{+}(x,y,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a^{+}(x,y,n)\exp(-inz)\,dn \qquad (38)$$

4. The Fourier transform denoted by a superscript bracket, thus

$$\hat{a}(x,m,n) = \iint_{-\infty}^{\infty} a(x,y,z)\exp\ i(my+nz)dy\ dz \qquad (39)$$

the solution to which is

$$a(x,y,z) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \hat{a}(x,m,n)\exp\ -i(my+nz)dm\ dn \qquad (40)$$

5. The Fourier transform

$$a^{*+}(\ell,m,n) = \iiint_{-\infty}^{\infty} a(x,y,z)\exp\ i(\ell x+my+nz)dx\ dy\ dz \qquad (41)$$

which can be inverted to give

$$\hat{a}(x,m,n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a^{*+}(\ell,m,n)\exp(-i\ell x)\ d \qquad (42)$$