SOME PROPERTIES OF THE PRODUCT OF SERIES INFINITE IN BOTH DIRECTIONS

THESIS

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By

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COURSES

THE STUDENT HAS PASSED THE FOLLOWING COURSES IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF M. SC.

1. Mathematical Analysis A	3 h. per week
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To My Parents

and

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CONTENTS

SUMMARY

This thesis deals with the ordinary convergence of series and Cesaro's method of summing divergent series. It also deals with the summability of the product of two series (infinite in one or both directions) by different rules of multiplication .

A new approach to this subject is introduced, i.e. the product of two series by squares, circles, rhombus and generely by curves . One of the series is numerical while the elements of the other belongs to a Banach space.

The thesis consists of three chapters. The first chapter contains some definitions on the series, Cesaro's method of summability and the definition of metric, linear normed spaces and Banach spaces.

In the second chapter we discuss Cauchy product of two series infinite in one direction and the classical theorems for multipliction by Cauchy's rule (Cauchy's theorem, Marten's theorem and Able's theorem).A version of these theorems,when one of the series has elements of a Banach space is given in a form of lemmas.

In the second part of this chapter, the method of multiplication of series by squares is concerned. the summability (by Cesa'ro) of the product by squares are discussed.

Chapter three deals with series infinite in both directions. We discuss diferent definitions of the sum of the series and study some methods of multiplication of the series. In §1, we discuss Laurant and Fourier methods of multiplication of series. The classical convergence theorems on the multiplication by these rules have been studied.

In §2, a new method of multiplication of series is defined, that is the method of multiplication of two series by curves. In particular, the curve may be a square or a circle or a rhombus.

Here, we prove that the product of two restrictedly convergent series by squares converges. Using this theorem, we prove that the product of absolutely convergent and restrictedly convergent series by squares, circles, curves and rhomus are equivalent. In all this cases, the resultant series converges. In the later case, we prove that the product of two absolutely convergent series is absolutely convergent and give an example showing that, this result is "exact" in some sense, viz. the product of two restrictedly convergent series may actually diverge.

In §3, we prove that, if one series is restrictedly convergent while the other is (c,1) and its sequance of partial sums is bounded, then their product by squares is (c,1) convergent.

In all the above mentioned cases one series is in $\mathbb{C}$ while the other is in a Banach space.

Finally, we prove that the product of two unrestrictedly convergent numerical series by rhombus is (c,1) convergent. Also, the summability of the product of two summable (by Cesa'ro) numerical series by squares investigated.

CHAPTER (I)

INTRODUCTION

Introduction:

As early as when studying the elementary courses of mathematical sums coı we came across sums consisting of an infinite number of summands (say, the sum of an infinite number of terms of a geometric progression). Sums of this kind are

Series of real and complex numbers: [4]

The sum of the infinite series $a_0+a_1+ \dots +a_n+ \dots$ is defined as

$$\lim_{n \to \infty} (a_0+ \dots +a_n)$$

Provided that the limit exists. This, however, is the definition of the sum of an infinite series and is not the definition of " infinite series " itself

Definition 1.1 :

The infinite series $\sum_{n=0}^{\infty} a_n$ is an ordered pair $\langle (a_n)_{n=0}^{\infty}, (s_n)_{n=0}^{\infty} \rangle$ where $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers and

$$s_n=a_0+a_1+ \dots +a_n \qquad (n \in N)$$

The number a_n is called the n^{th} term of the series and s_n is called the n^{th} partial sum of the series.

The definition of convergence or divergence of the series $\sum_{n=0}^{\infty} a_n$ depends on the convergence or divergence of the sequence $(s_n)_{n=0}^{\infty}$ of partial sums.

Definition 1.2 :

Let $\sum_{n=0}^{\infty} a_n$ be a series of real numbers with partial sums $s_n = a_0 + \ldots + a_n$ $(n \in \mathbb{N})$. If the sequence $(s_n)_{n=0}^{\infty}$ converges to $A \in \mathbb{R}$, we say that the series $\sum_{n=0}^{\infty} a_n$ converges to A. If $(s_n)_{n=0}^{\infty}$ diverges we say that the $\sum_{n=0}^{\infty} a_n$ diverges. If the series $\sum_{n=0}^{\infty} a_n$ converges to A, we often write $\sum_{n=0}^{\infty} a_n = A$. We sometimes call the series $\sum_{n=0}^{\infty} a_n$ series infinite in one direction.

Summability of series: [4],[5]

One important branch of the field of infinite series is the study of summability of divergent series. The theory of summability has many uses throughout analysis and applied mathematics. This study is an attempt to attach a value to series that may not converge- that is, an attempt to generalize the concept of the sum of a convergent series. Many (but not all) of the well-known methods of summability deal exclusively with the sequence of partial sums of an infinite series.

An adopted method for summability is the following :

We apply the transformation (T) to the sequence of partial sums A_n $(A_n = \sum_{k=0}^{n} a_k)$ of the series $\sum_{n=0}^{\infty} a_n$. Let $t_n = T(A_n)$ The transformation (T) may be a function of a continuous variable ($t(x)=T(A_n)$).

Then we study the ordinary convergence of the newly obtained sequence or function. We say that the given series is summable by the method T if the transformed sequence or function tends to a limit.

Definition 1.3 :

We say that $\lim_{n \to \infty} A_n = L(T)$ if and only if

$$t_n \to L \quad \text{as} \quad n \to \infty \quad (\text{or } t(x) \to L \quad \text{as} \quad x \to 1)$$

Cesa'ro's method of summability :

The idea of Cesa'ro method is to take the arithmetic mean of the terms comprising the sequence and study the convergence of these means.

First we define the (c,1) method and then define the (c,k) method of summation.

1. The (c,1) method [4]

Just as the convergence of the series $\sum_{n=0}^{\infty} a_n$ is defined to mean the convergence of the sequence $(s_n)_{n=0}^{\infty}$ of its partial sums,the (c,1)summability of $\sum_{n=0}^{\infty} a_n$ will now be defined to mean the (c,1) summability of $(s_n)_{n=0}^{\infty}$.

Definition 1.4 :

We say that the sequence $(s_n)_{n=0}^{\infty}$ is (c,1) summable to s if

$$\lim_{n \to \infty} \sigma_n = s$$

where

$$\sigma_n = \frac{s_0+s_1+s_2+ \dots +s_n}{n+1}$$

Definition 1.5 :

Let $\sum_{n=0}^{\infty} a_n$ be a series of real numbers with partial sums $s_n = a_0+a_1+ \dots +a_n$

We shall say that $\sum_{n=0}^{\infty} a_n$ is (c,1) summable to s if

$$\lim_{n \to \infty} s_n = s \quad (c,1)$$

In this case we write

$$\sum_{n=0}^{\infty} a_n = s \qquad (c,1)$$

Definition 1.6 :

A method T of summability for series is called regular if every convergent series is T summable to its sum.