SYSTEMS OF INTEGRAL EQUATIONS OF VOLTERRA TYPE

A THESIS

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بسم الله الرحمن الرحيم

CONTENTS

SUMMARY

The aim of the present thesis is:

(I) The study of the theory of existence and uniqueness of solution of a system of integral equations of volterra type.

(II) The stady of some special types of such equations.

The thesis consists of three chapters:

The first chapter serves as an introduction. It contains some basic concepts from mathematical analysis. These concepts are used in chapter II and III.

In § 1.1 we have studied some analytic properties of the spaces $C^{(n)}(a,b)$ and $L_2^{(n)}(a,b)$; like completness of $C^{(n)}(a,b)$ and completness of $L_2^{(n)}(a,b)$.

In § 1.2 we have studied Laplace's transform and the related convolution theorem.

§ 1.3 deals with Fubini's theorem stating that; if f (x,y) is a function defined in the rectangle $P = \left\{ a \leqslant x \leqslant b \quad , \quad c \leqslant y \leqslant d \right\}$ for which the double integral

$$\iint_P f(x,y)\, d\, x\, d\, y \qquad \text{(I)}$$

exists and onefold integral

$$J(x) = \int_c^d f(x,y)\, dy \qquad \text{(II)}$$

exists for fixed value of x in the interval $a \leqslant x \leqslant b$, then the iterated integral

$$\int_a^b dx \int_c^d f(x,y)\, dy = \int_a^b J(x)\, dx \qquad (III)$$

also exists and we have

$$\iint_P f(x,y)\, dx\,dy = \int_a^b dx \int_c^d f(x,y)\, dy.$$

And if $J_1(y) = \int_a^b f(x,y)\, dx$ exists,

then

$$\iint_P f(x,y)\, d\,x\, d\,y = \int_a^b dx \int_c^d f(x,y)\, dy =$$

$$= \int_c^d dy \int_a^b f(x,y)\, dx$$

The second chapter is the main part of the thesis where we have concentrated on the existence and uniqueness of solution of the problem

$$\Phi(x) - \lambda \int_0^x K(x,y)\, \Phi(y)\, dy = F(x) \quad , \qquad (IV)$$

where

$$K(x,y) = (k_{ij}(x,y)) \; ; \quad i,j = 1,2, \ldots, n,$$

$$F(x) = (f_i(x)) \qquad ; \quad i = 1,2,\ldots, n,$$

and λ is a parameter (in general a complex parameter). This, in fact, is a generalization of the theory of existence and uniqueness of a solution of one volterra integral equation.

§ 2.1 deals with the formal theory of existence.

§ 2.2 discusses the invarience of $L_2^{(n)}$ with respect to $\tilde{K}$. We have proved that if $G(x) \in L_2^{(n)}$, and $K(x,y) \in L_2^{(n^2)}$, then

$$(\tilde{K}\, G)(x) = \int_0^x K(x,y)\, G(y)\, dy = \Psi(x)$$

belongs also to $L_2^{(n)}$, i.e. the operator $\tilde{K}$ maps $L_2^{(n)}$ into itself.

In § 2.3 we have proved that the system of integral equations IV has a solution in the form

$$\Phi(x) = F(x) - \lambda \int_0^x H(x,y;\lambda)\, F(y)\, dy \qquad (VI)$$

where $H(x,y;\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,y)$

is convergent for all λ and $K_{n+1}(x,y)$ is the iterated matrix - kernel of the matrix - kernel $K(x,y)$.
In § 2.4 it is proved that problem (IV) has a unique solution $\Phi(x)$ having the form (VI).

In the third chapter we studied some special types of systems of Volterra integral equations.

In § 3.1 we have proved that the Laplace's transform may be employed in the solution of a system of Volterra integral equations of the convolution type;

$$\phi_i(x) = \sum_{j=1}^{n} \int_0^x k_{ij}(x-y)\, \phi_j(y)\, dy + f_i(x),$$

$$i = 1, 2, \ldots, n$$

where the kernel $(k_{ij}(x-y))$; $i, j = 1, 2, \ldots, n$ is dependent of the difference $(x-y)$ and $k_{ij}(x)$, $f_i(x)$ are known continuous functions having Laplace's transforms.
§ 3.2 studies the system of Abel's type; we have proved that the system $\int_0^x \frac{A\,\Phi(y)}{(x-y)^{\alpha}}\, dy = F(x)$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ is a constant invertible matrix,

$$\Phi(y) = \begin{bmatrix} \phi_1(y) \\ \vdots \\ \phi_n(y) \end{bmatrix}$$ is the unknown vector and $F(x) = \left(f_i(x)\right)$

$i = 1, 2, \ldots, n$, is a given vector, can be solved.

The thesis is ended by an appendix showing the connection between systems of ordinary differential equations with initial values and systems of Volterra integral equations.

CHAPTER I

PRELIMINARIES AND INTRODUCTION

CHAPTER I

Preliminaries and introduction

§ 1.1. Some vector and matrix functional spaces :

In this paragraph we introduce some normed spaces, the elements of which are vectors of functions or matrices of functions. These spaces are useful in the study of the theory of systems of integral equations of Volterra type which is the principal aim of the present thesis.

The spaces $C^{(n)}$ (a,b) and $C^{(n^2)}(a,b)x$ (a,b) :

The elements of the space $C^{(n)}(a,b)$ are all vector - functions consisting of n components, each of them is a continuous function defined on the interval (a,b) :

$$
\begin{aligned}
&F(x) \in C^{(n)}(a,b) \iff \\
&F(x) = \left\{ f_1(x), f_2(x), \ldots, f_n(x) \right\} \quad \text{such that} \\
&f_i(x) \in C(a,b) \qquad \forall i = 1,2,\ldots,n \quad ,
\end{aligned}
$$

where C (a,b) is the spaces of all continuous functions defined on the interval (a,b) [4].

The sapce $C^{(n)}(a,b)$ can be made a linear space [7,8,9] by introducting the following linear operations. If,

$$F(x) = \left\{ f_1(x),\ f_2(x),\ \ldots,\ f_n(x) \right\},$$

$$G(x) = \left\{ g_1(x),\ g_2(x),\ \ldots,\ g_n(x) \right\},$$

$\lambda \in \mathbb{C}$ ($\mathbb{C}$ is the field of complex numbers).

Then

$$(F+G)(x) = \left\{ f_1(x)+g_1(x),\ f_2(x)+g_2(x),\ \ldots,\ f_n(x)+g_n(x) \right\},$$

$$(\lambda F)(x) = \left\{ \lambda f_1(x),\ \lambda f_2(x),\ \ldots,\ \lambda f_n(x) \right\}.$$

It is quite easy to verify that all the axioms of the linear space [7] are satisfied.

To make $C^{(n)}(a,b)$ a normed space, we introduce one of the following norms;

$$\left.\begin{array}{l} \|F\|^{(1)}_{C^{(n)}} = \max\limits_{i} \left\{ \sup\limits_{a<x<b} |f_i(x)| \right\} \quad (\mathrm{I}) \\ \qquad\qquad\qquad\qquad \forall i = 1,2,\ldots,n \\ \text{or} \\ \|F\|^{(2)}_{C^{(n)}} = \sum\limits_{i=1}^{n} \sup\limits_{a<x<b} |f_i(x)| \quad . (\mathrm{II}) \end{array}\right\} \quad (1.1.1)$$

The fact that they satisfy the properties of the norm is an easy task.

Definition 1 [5]: (1)

In a linear space E we say that the two norms $\| \|^{(1)}$ and $\| \|^{(2)}$ are equivalent (or define the same topology) if, and only if, there exist two nonnegative constants $\varepsilon_1, \varepsilon_2$ such that

$$\varepsilon_2 \|x\|^{(2)} \leqslant \|x\|^{(1)} \leqslant \varepsilon_1 \|x\|^{(2)} \quad \forall x \in E.$$

Proposition 1:

The two norms defined in (1.1.1) are equivalent.

Proof

We shall prove that there exists $\varepsilon_1, \varepsilon_2$, such that

$$\varepsilon_2 \|F(x)\|^{(2)} \leqslant \|F\|^{(1)} \leqslant \varepsilon_1 \|F(x)\|^{(2)}.$$

Since

$$\|F\|^{(1)}_{C^n(a,b)} = \max_i \left\{ \sup_{a<x<b} |f_i(x)| \right\}, \quad \forall i=1,2,\ldots,n.$$

Then

$$\sup_{a<x<b} |f_i(x)| \leqslant \|F\|^{(1)}.$$

By summing from i = 1 up to i = n, we get

$$\sum_{i=1}^{n} \sup_{a<x<b} |f_i(x)| \leqslant n \, \|F\|^{(1)} \quad \forall i$$

i.e. $\quad \frac{1}{n} \|F\|^{(2)} \leqslant \|F\|^{(1)}.$

Hence, taking $\varepsilon_2 = \frac{1}{n}$, we have

$$\varepsilon_2 \|F\|^{(2)} \leqslant \|F\|^{(1)}. \tag{1.1.2}$$

Secondly, since

$$\|F\|^{(2)}_{C^{(n)}(a,b)} = \sum_{i=1}^{n} \sup_{a<x<b} |f_i(x)|.$$

Hence;

$$\sup_{a<x<b} |f_i(x)| \leqslant \|F\|^{(2)} \quad \forall i. \tag{1.1.3}$$

By taking the maximum on both sides of (1.1.3), we get

$$\max \left\{ \sup_{a<x<b} |f_i(x)| \right\} \leqslant \|F\|^{(2)},$$

i.e.

$$\|F\|^{(1)} \leqslant \|F\|^{(2)} \qquad \text{(here } \varepsilon_1 = 1\text{).} \tag{1.1.4}$$

From (1.1.2) and (1.1.4), we get

$$\varepsilon_2 \|F\|^{(2)} \leq \|F\|^{(1)} \leq \|F\|^{(2)} .$$

This proves that $\|F\|^{(1)}_{C^{(n)}_{(a,b)}}$ and $\|F\|^{(2)}_{C^{(n)}_{(a,b)}}$ are equivalent.

Proposition 2:

$C^{(n)}$ (a,b) is a complete normed space (or, in other words, a Banch space).

Proof

We shall prove that each fundamental sequence in $C^{(n)}(a,b)$ has a limit in $C^{(n)}(a,b)$. Let $F_m(x) = \left\{ f_1^{(m)}(x), f_2^{(m)}(x), \ldots, f_n^{(m)}(x) \right\}$ be fundamental sequence in $C^{(n)}$ (a,b); i.e. for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m > N$, $\ell > N$,

$$\| F_\ell (x) - F_m (x) \|_{C^{(n)}(a,b)} = \max \| f_i^{(\ell)} - f_i^{(m)} \| < \varepsilon$$

i.e.

$\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m > N$, $\ell > N$,