NUMERICAL SOLUTION OF INITIAL BOUNDARY VALUE PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS

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BY

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بسم الله الرحمن الرحيم

CONTENTS

SUMMARY

In this thesis we adopt the finite difference methods for solving initial - boundary value problems for partial differential equation is an approximate method , in the sense that derivatives at a point are approximated by difference quotients over a small interval. This method reduces the problem to difference equations forming a system of algebraic equations of the solution at different points. We apply a finite difference method as Wendroff's and modified Wendroff's methods for solving the initial - boundary value problem of hyperbolic first order partial differential equation in different spatial spaces.

In chapter I : We introduced the concepts form of the theory of partial differential equations such as types of the initial and boundary conditions, the classification of partial differential equation for first and second - order equations and Well - posed problems.

In chapter II : We introduced the concepts of the finite difference method, and we have introduced a survey of the difference equations imposed for the heat and wave equations. We have studied the errors associated with finite differences techniques, the consistency and convergence . For studying the stability of a difference equation, it is considered the Von Neumann method.

In chapter III : We have studied Wendroff's and modified Wendroff's methods as a finite difference approximation for solving the initial boundary value problem of hyperbolic first order partial differential equation in different spatial spaces and we discuss the

these methods. Also we introduced a comparison between these methods by using numerical examples. and we apply the Wendroff's and modified Wendroff's methods for solving initial - boundary value problems for hyperbolic system of partial differential equations with the study of the stability. Numerical examples are done for justifying the results.

CHAPTER I

INITIAL - BOUNDARY VALUE PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS.

In this chapter we introduced the concepts form of the theory of partial differential equations such as types of the initial and boundary conditions, the classification of partial differential equation for first and second-order equations and well-posed problems.

1.1 Initial - Boundary Value Problems :

If the equation contains time derivatives up to order K , the initial state can be characterized by specifying the initial values of the unknown function and its time derivatives through order K - 1 , and the initial - boundary value problem depending on the time and the boundary in the spatial space .

Initial value problem as :

$$u_{tt} = a^2 u_{xx} \qquad \text{in } R^n ,\ t > o$$

$$u(x,o) = f(x) \qquad \text{in } R^n$$

$$u_t(x,o) = g(x) \qquad \text{in } R^n$$

Initial - boundary value problem as :

$$u_t = a\, u_{xx} \qquad \text{in } \Omega ,\ t \geq o$$

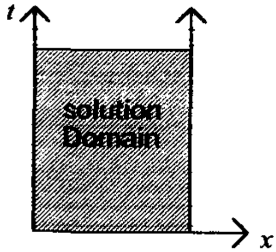
$$u(x,o) = f(x)$$

$$u(o,t) = g_0(x) \qquad ,\ t \geq o$$

$$u(a,t) = g_1(x) \qquad ,\ t \geq o$$

Given the description of $f(x)$, $g_0(t)$ and $g_1(t)$, the solution $u(x,t)$ is to be determined over the semi - infinite rectangle in x - t space defined by $o \leq x \leq a$, $t \geq o$ Figure (1.1).

Boundary condition $g_0(t)$ defined on $x=o$
i.e.
$u(o,t) = g_0(t), t \geq o$



Boundary condition $g_1(t)$ defined on $x=a$
i.e.
$u(a,t) = g_1(t), t \geq o$

Figure (1.1)

Initial condition $f(x)$
defined at $t = o$
i.e. $u(x,o) = f(x)$, $o \leq x \leq a$

The region Ω on which the solution of the given differential equation $u_t = \alpha\, u_{xx}$ is to be found subject to the initial and boundary condition , is called the solution domain.

1.2 Initial and Boundary Conditions :

The auxiliary conditions, which must be imposed on a PDE fall into categories initial and boundary conditions. The term "initial" in PDE refers to the fact in any physical problems one of the independent variables, say x , may be the time.

Thus the unknown function $u(x,y)$ at $x = 0$ is referred to as an initial condition, (Lapidus[8] & Jain [5]). In general, initial and boundary conditions have the form :

$$\alpha(x,y)\, u(x,y) + \beta(x,y)\, u_n(x,y) = \gamma(x,y) \qquad (1.1)$$

Where $u_n(x,y) = \frac{\partial u}{\partial n}$ is considered as the derivative normal to the boundary for y belongs to the points at the boundary

Frequently $\frac{\partial u}{\partial n} = u_x$ or u_y, the condition (1.1) can be classified as homogeneous $(\gamma = 0)$ or non homogeneous $(\gamma \neq 0)$.

The three main types of boundary conditions (BCs) in the boundary value problems can be summarized as follows :

1. ***BCs of the first kind (Dirichelt BCs):***

Where the PDE holds over a given region of space and the solution is specified on the boundary of the region i.e $\beta = 0$ in equation (1.1).

2. ***BCs of the second kind (Neumann BCs):***

Where the PDE holds in some region of space but now the outward normal derivative $\frac{\partial u}{\partial n}$ (which is proportional to the inward flux) is specified on the boundary. i.e $(\alpha = 0)$ in equation (1.1).

3. ***BCs of the third kind (Robin BCs):***

These problems correspond to PDEs being given in some region of space, but now the condition on the boundary is a mixture of the first two kinds i.e. $(\alpha \neq 0)$ and $(\beta \neq 0)$ in equation (1.1) the combination of PDE initial and boundary condition must lead to a well-posed problem i.e. if its solution exists, is unique and depends continuously on the given data and depending on the form of the x, y region.

1.3 Classification:

1.3.1 Types of First - Order Equations :

(a) Linear first order :

If the coefficients are constants or functions of the independent variables only, the PDE is linear , for example

$$u_t + a\,u_x = b$$

where a and b are constants or functions of the independent variables.

(b) Quasilinear first order :

If the coefficients are also functions of the dependent variable the PDE is quasilinear, for example

$$u_t + u\,u_x = t^2$$

(c) Nonlinear first order :

If the coefficients are functions of the first derivatives the PDE is nonlinear , for example

$$u_t + (u_x)^2 = 0$$

In general, when the coefficient of an nth-order PDE depend upon the nth-order derivatives, the equation is nonlinear; when they depend upon mth-order derivatives where $m < n$, the equation is quasilinear, (Lapidus [8] and Williams [22]).

The general quasilinear system of n first - order PDEs in functions of two independent variables x , y :

$$\sum_{j=1}^{n} a_{ij}\frac{\partial u_j}{\partial x} + \sum_{j=1}^{n} b_{ij}\frac{\partial u_j}{\partial y} = c_i \qquad (i = 1,2,\dots,n) \qquad (1.2)$$

Where a_{ij}, b_{ij} and c_i may depend on x , y , u_1 , u_2, ..., u_n. If each a_{ij} and b_{ij} is independent of u_1 , u_2, ..., u_n , the system (1.2) is called almost linear. In addition , if each c_i depends linearly on u_1 , u_2, ..., u_n , the system is said to be linear.

In terms of $n \times n$ matrices $A = [a_{ij}]$, and the column vectors $u=(u_1 ,u_2 , \dots, u_n)^T$ and $c=(c_1 ,c_2, \dots, c_n)^T$, the system (1.2) can be expressed as:

$$Au_x + B\,u_y = c \qquad (1.3)$$

If A or B is nonsingular , it is usually possible to classify system (1.3) according to type. Suppose that $det(B) \neq 0$ and define a polynomial of degree n in λ by

$$P_n(\lambda) = det(A^T - \lambda B^T) = det(A - \lambda B) \qquad (1.4)$$

system (1.3) is classified as:

<u>Elliptic</u> : If $P_n(\lambda)$ has no real zeros.

Hyperbolic : If $P_n(\lambda)$ has n real , distinct zeros, or if $P_n(\lambda)$ has n real zeros,at least one of which is repeated, and the generalized eigenvalue problem $(A^T - \lambda B^T)t = 0$ yields n linearly independent eigenvectors t .

Parabolic : If $P_n(\lambda)$ has n real zeros, at least one of which is repeated, and the above generalized eigenvalue problem yields fewer than n linearly independent eigenvectors.

An exhaustive classification cannot be carried out when $P_n(\lambda)$ has both real and complex zeros.

Example 1 :

If the Cauchy - Riemann equations, $u_x = v_y, u_y = -v_x$ are written in the form (1.3) , then

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and $det(A - \lambda B) = 1 + \lambda^2$ which has no real zeros. Thus the Cauchy Riemann equations are elliptic .

Example 2 :

Show that the system

$$2u_x - 2v_y + u_y - 3v_y = v \ ,$$

$$u_x - 4v_x + v_y = u$$

is hyperbolic

Writing this system in the form (1.3) we have

$$A = \begin{bmatrix} 2 & -2 \\ 1 & -4 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

and $det(A - \lambda B) = \lambda^2 - \lambda - 6$ has distinct real zeros, $\lambda_1 = 3$ and $\lambda_2 = -2$, the system is hyperbolic

1.3.2 Types of Second - Order Equations :

(a) The linear PDE of the second order in two independent variables :

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \qquad (1.4)$$

where a, b, c, d, e, f and g may be constants or functions of x and y.

Equation (1.4) is classified as :

Hyperbolic if $b^2 - 4ac > 0$

Parabolic if $b^2 - 4ac = 0$

Elliptic if $b^2 - 4ac < 0$

(Stephenson [19] and Meis [9]).

Note that :

The type of (1.4) is determined by its principal (the terms involving the highest-order derivatives of u), and that the type will generally change with position in the xy - plane unless a, b, and c are constants .

(b) The general linear PDE of order two in n variables :

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = g \qquad (1.5)$$

If $u_{x_i x_j} = u_{x_j x_i}$, then the principal part of (1.4) can always be arranged so that $(a_{ij} = a_{ji})$; thus the $n \times n$ matrix $A = [a_{ij}]$ can be assumed symmetric. Since every real symmetric $n \times n$ matrix has n real eigenvalues these eigenvalues are the (possibly repeated) zeros of an n th-degree polynomial in λ ;

$$\det (A - \lambda I) = 0$$

where I is the $n \times n$ identity matrix .

Let P denote the number of positive eigenvalues of A, and Z the number of zero eigenvalues of A (i.e. the multiplicity of the eigenvalue zero) then equation (1.5) is

Parabolic	:	If $Z > 0$ (equivalently if $\det A = 0$)
Elliptic	:	If $Z > 0$ and $P = n$
	Or	$Z = 0$ and $P = 0$
Hyperbolic	:	If $Z = 0$ and $P = 1$
	Or	$Z = 0$ and $P = n - 1$
Ultrahyperbolic	:	If $Z = 0$ and $1 < P < n - 1$

We note that, in this case n must be greater than three. (Courant [3])

Example 1 :

For the PDE $3u_{x_1x_1} + u_{x_2x_2} + 4u_{x_2x_3} + 4u_{x_3x_3} = 0$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \text{ and } \det(A - \lambda I) = \lambda(3-\lambda)(\lambda-5)$$

Since $\lambda = 0$ (i.e $Z = 1 > 0$) then the PDE is parabolic.

Example 2 :

For the PDE $u_{xx} + u_{xy} + 5u_{yx} + u_{yy} + 2u_{yz} + u_{zz} = 0$

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } \det(A-\lambda I) = (1-\lambda)(\lambda^2 - 2\lambda - 9)$$

Then the eignvalues of A are 1 and $1 \pm \sqrt{10}$ thus, $Z = 0$ and $P=3-1$, making the PDE hyperbolic.

1.4 Well - posed problems :

The prescribed initial and boundary condition functions, together with the coefficient functions and any inhomogeneous term in the PDE, are said to comprise the data in the problem modeled by the PDE. The solution is said to depend continuously on the data if small changes in the data produce corresponding small changes in the solution, the problem itself is said to be well-posed if :

(i) A solution to the problem exists.

(ii) The solution is unique..