Ain Shams University
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SOME TOPICS IN PARTIAL DIFFERENTIAL EQUATIONS

THESIS

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List of symbols

R^n – n-dimensional Euclidean space of points $x = (x_1, \ldots, x_n)$.

$$|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \quad x^2 = |x|^2$$

$$u_{x_i} = \frac{\partial u}{\partial x_i}, \quad u_t = \frac{\partial u}{\partial t}, \quad u_{x_i x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}$$

$$u_x = (u_{x_1}, \ldots, u_{x_n}), \quad |u_x|^2 = \sum_{i=1}^{n} u_{x_i}^2$$

$$\frac{\partial^k u}{\partial x^k} = \frac{\partial^k u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}, \quad \text{where } k = k_1 + \ldots + k_n.$$

$$\sum_k \left(\frac{\partial^k u}{\partial x^k}\right)^2 = \sum_{k_1+\ldots+k_n=k} \left(\frac{\partial^k u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}\right)^2$$

$C^k(\Omega)$ – the class of functions having continuous derivatives up to order k.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{– Kronecher's symbol.}$$

CONTENTS

INTRODUCTION AND SUMMARY

Boundary value problems in domains with smooth boundaries for linear partial differential equations are well studied. The existence and the uniqueness of the solutions of such problems in different functional spaces were proved in [1], [2] and others. Moreover, it was shown that a better smoothness of the functions defining the problem (i.e. the functions which are considered known, such as the coefficients, the right hand side of the equation, the initial and boundary conditions) yields better differential properties of the solution.

Boundary value problems for elliptic equations in plane domains with angles on its boundaries were studied in [6], [7], and others.

In [8], and [9] the general elliptic problems in n-dimensional domains with conic points on the boundaries, and also the smoothness of the solution of the first boundary value problem near an edge on the boundary were studied. Analogous results for equations of parabolic type in domains with sectionally smooth boundaries were obtained by M.I. Hassan [10]. He also obtained the main term of the asymptote of the solution of the first boundary value problem for parabolic equation near an edge.

This thesis aims at investigating the main results concerning the existence, uniqueness and smoothness of the solutions belonging to Sobolev spaces of the elliptic and parabolic equations in domains with smooth and sectionally smooth boundaries. It also aims at obtaining the singularities of the solution of the first boundary value problem for elliptic equation near an edge. It will be shown that such a solution is smooth everywhere in the domain, except at the edge, near which it behaves as a power of the distance from this edge. This is done by using special spaces of functions having derivatives, integrable with some weight.

The first chapter is devoted to a study of Sobolev spaces of positive integral order and their basic properties, including the very important Sobolev's imbedding theorems.

In chapter II we consider the equation

$$A\,u = -\sum_{i,k=1} (a_{ik}\, u_{x_k})_{x_i} + c\,u = f \qquad (1)$$

with real coefficients such that $a_{ik} = a_{ki} \in \bar{C}^1(\Omega)$, where Ω is some domain in R^n, $c(x) \in C(\Omega)$ and $c(x) \geqslant 0$. We assume that (1) is an elliptic equation, i.e. that, for $x \in \Omega$ and all real $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$,

$$c^2 \left|\xi\right|^2 \leqslant \sum_{i,k=1}^{n} a_{ik}(x)\, \xi_i\, \xi_k \leqslant C^2 \left|\xi\right|^2 \qquad (2)$$

for some constants c and C.

A generalized solution $u(x) \in W^1(\Omega)$ of the first boundary value problem for (1) is defined to be a function of $W^1(\Omega)$ satisfying the integral identity

$$\iint_{\Omega} \left(\sum_{i,k=1}^{n} a_{ik} u_{x_k} v_{x_i} + c\, u\, v \right) dx = \iint_{\Omega} f\, v\, dx \qquad (3)$$

for all $v \in {}^{o}W^1(\Omega)$ [it is assumed that $f \in L_2(\Omega)$] and the boundary condition

$$u\big|_{\Gamma} = \Phi \qquad (4)$$

where Φ is a given function on the boundary Γ of Ω.

The main result concerning problem (1), (4) is that it has a unique solution in $W^1(\Omega)$ only if Φ can be continued into Ω, i.e. it is necessary that there be a function $\Psi \in W^1(\Omega)$, whose trace on Γ is Φ : $\Psi\big|_{\Gamma} = \Phi$.

In chapter II we consider also the solution of the mixed boundary value problem for the parabolic equation

$$\frac{\partial u}{\partial t} + A\, u = f(x, t)$$

where A is an elliptic operator in the cylinder

$$Q = \Omega \times (0,T).$$

The scheme of Fourier's method is studied for this problem.

In chapter III we study solutions of the first boundary value problem for the elliptic equation of the general form

$$L u = \sum_{i,k=1}^{n} a_{ik}(x)\, u_{x_i x_k} + \sum_{i=1}^{n} a_i(x)\, u_{x_i} + a(x)\, u = f(x) \quad (5)$$

with real infinitely differentiable coefficients in a bounded region Ω, whose boundary Γ is piecewise-smooth surface formed of (n-1) dimensional infinitely smooth surfaces S_i $(i = 1,2, \ldots, m)$. We also assume that S_i intersects only S_{i-1} and S_{i+1} along smooth (n-2)-dimensional manifolds ℓ_{i-1} and ℓ_{i+1}.

We shall consider in detail the case in which m = 2, where the obtained results are of local nature.

The results of this chapter are formulated in terms of the angle $\omega(p)$ which is defined as follows:

Let R_1 and R_2 be the plans tangent to S_1 and S_2 respetively at the point $P \in S_1 \cap S_2$. We reduce the equation

$$\sum_{i,k=1}^{n} a_{ik}(P)\, u_{x_i x_k} \equiv 0$$

to canonical form. The reduction transforms the planes R_1 and R_2 into other planes R_1' and R_2'. The angle between these latter planes is denoted by $\omega(P)$. The solution of (5) is assumed to be vanishing on Γ and belonging to $W^1(\Omega)$.

Certain function spaces will be used in chapter III. $W_\alpha^k(\Omega)$ is the space with norm

$$\| u \|^2_{W^k_\alpha(\Omega)} = \iint_\Omega \left[\rho^\alpha \left| \frac{\partial^k u}{\partial x^k} \right|^2 + | u |^2 \right] dx < \infty$$

where $\rho(x)$ is everywhere infinitely differentiable function and positive except on $\Gamma_o = S_1 \cap S_2$, and coincide in some neighbourhood of Γ_o with the distance $r(x, \Gamma_o)$ from x to Γ_o.

${}^oW^k_\alpha(\Omega)$ is the space with the norm

$$\| u \|_{{}^oW^k_\alpha(\Omega)} = \sum_{s=0}^{k} \iint_\Omega \rho^{\alpha+2s-2k} \left| \frac{\partial^s u}{\partial x^s} \right|^2 dx < \infty .$$

We shall discuss the results about the smoothness of the solution in Ω. It will be shown that the solution of $Lu = f$, where $f \in W^k_\alpha(\Omega)$, belongs to $W^{k+2}_\alpha(\Omega)$ if the angle ω is sufficiently small, namely, ω must satisfy the inequality

$$\frac{2\pi}{\omega} > 2k + 2 - \alpha$$

To study the singularities of this solution near the edge Γ_o, several properties of the spaces $W^k_\alpha(\Omega)$ and ${}^oW^k_\alpha(\Omega)$ are proved. The proofs are done in similar ways to those used in [10].

CHAPTER I

SOBOLEV SPACES.

I.1 Introduction:

In this chapter, we shall introduce and develop all the basic properties of Sobolev spaces of positive integral order. In section 2, we concentrate on the Lebesgue spaces $L_p(\Omega)$, of which Sobolev spaces are special subspaces. Section 3, is devoted to a study of generalized functions and weak derivatives. In section 4, we shall restrict our attention to the main purpose of this chapter, which is the study of properties of Sobolev spaces.

I.2. The space $L_p(\Omega)$

Let Ω be a domain in the Euclidean space R^n of points $x = (x_1, \ldots, x_n)$ and p be a positive real number. We denote by $L_p(\Omega)$, the Banach space of all functions on Ω that are measurable, and that are p-summable with respect to Ω with $1 \leq p < \infty$. The norm in this space is defined by

$$\| u \|_p^p = \int_\Omega | u |^p \, dx$$

The elements of this space are classes of equivalent functions on Ω, since two functions are identified if they differ only on a set of n-dimensional measure zero.

A function u, measurable on Ω, is said to be essentially bounded on Ω, if there exists a constant $K > 0$ for which $u(x) \leq K$ almost everywhere on Ω. The greatest lower bound of such constants K is called the essential supremum of $|u|$ on Ω, and is denoted by $\operatorname{ess\,sup}_{x \in \Omega} |u|$. We denote by $L_\infty(\Omega)$, the Banach space of essentially bounded functions u on Ω with norm

$$\|u\|_\infty = \operatorname{ess}\sup_{x \in \Omega} |u| .$$

We shall frequently use the following well-known algebraic and functional inequalities:

(1) Cauchy's inequality

$$a\,b \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2$$

which is valid for arbitrary $\epsilon > 0$ and arbitrary a and b.

(2) Minkowski's inequality

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p$$

which is valid when $1 \leq p < \infty$.

(3) Hölder's inequality

If $1 < p < \infty$ and $u \in L_p(\Omega)$, $v \in L_{p'}(\Omega)$, where $p' = \frac{p}{p-1}$ is the exponent conjugate to p, then $uv \in L_1(\Omega)$ and

$$\int_\Omega |u\,v|\,dx \leq \|u\|_p \, \|v\|_{p'}$$

It is clear that $\frac{1}{p} + \frac{1}{p'} = 1$. This latter inequality extends to cover the two cases $p = 1$, $p' = \infty$ and $p = \infty$, $p' = 1$.

(4) Schwartz inequality

$$\int_{\Omega} |uv| dx \leqslant \left(\int_{\Omega} |u|^2 dx \right)^{1/2} \left(\int_{\Omega} |v|^2 dx \right)^{1/2}$$

is obtained from Hölder's inequality when $p = p' = 2$.

Imbedding theorem for L_p-spaces:

Let X, Y be normed spaces, We say that X is imbedded in Y, and write $X \longrightarrow Y$ to designate this imbedding, provided

(i) X is a subspace of Y

(ii) there exists a constant M such that

$$\| Ix \|_Y \leqslant M \| x \|_X \quad \forall x \in X$$

where I is the identity operator defined on X into Y by $Ix = x$ for all $x \in X$.

Sometimes the requirement that $X \subset Y$ and I be the identity map, is weakened to allow imbeddings of Sobolev spaces into spaces of continuous functions, and trace imbeddings of these spaces.

Now we formulate and prove a very useful imbedding result, for $L_p(\Omega)$ spaces and some of its consequences.

<u>Theorem (I.2.1)</u> If vol $\Omega = \int_\Omega dx < \infty$ and $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, then $L_{p_2}(\Omega) \longrightarrow L_{p_1}(\Omega)$.

If $u \in L_\infty(\Omega)$, then

$$\lim_{p_1 \to \infty} \|u\|_{p_1} = \|u\|_\infty \tag{I.2.1}$$

If $u \in L_{p_1}(\Omega)$ for $1 \leqslant p_1 < \infty$ and if there is a constant K such that for all such p_1

$$\|u\|_{p_1} \leqslant K, \tag{I.2.2}$$

then $u \in L_\infty(\Omega)$ and

$$\|u\|_\infty \leqslant K \tag{I.2.3}$$

In fact, Hölder's inequality gives

$$\|u\|_{p_1}^{p_1} = \int_\Omega |u|^{p_1} dx \leqslant \left\{ \int_\Omega |u|^{p_2} dx \right\}^{p_1/p_2} \left\{ \int_\Omega 1 \, dx \right\}^{1-(p_1/p_2)}$$

$$= (\mathrm{vol}\ \Omega)^{1-(p_1/p_2)} \|u\|_{p_2}^{p_1}$$

Hence

$$\|u\|_{p_1} \leqslant (\mathrm{vol}\ \Omega)^{\frac{1}{p_1} - \frac{1}{p_2}} \|u\|_{p_2} \tag{I.2.4}$$

and the first part of the theorem is proved.

If $u \in L_\infty(\Omega)$, we obtain from (I.2.4)

$$\lim_{p_1 \to \infty} \sup \|u\|_{p_1} \leqslant \|u\|_\infty \tag{I.2.5}$$

On the other hand, for any $\varepsilon > 0$ there exists a set $A \subset \Omega$ having positive measure $\mu(A)$ such that

$$|u(x)| \geqslant \|u\|_{\Omega} - \varepsilon \qquad \text{if} \quad x \in A$$

Hence,

$$\int_{\Omega} |u(x)|^{p_1}\, dx \geqslant \int_{A} |u(x)|^{p_1}\, dx \geqslant \mu(A)\, (\|u\|_{\infty} - \varepsilon)^{p_1}$$

It follows that

$$\|u\|_{p_1} \geqslant \left[\mu(A)\right]^{1/p_1} (\|u\|_{\infty} - \varepsilon), \text{ whence}$$

$$\lim_{p_1 \to \infty} \inf \|u\|_{p_1} \geqslant \|u\|_{\infty} \qquad (I.2.6)$$

Equation (I.2.2) now follows from (I.2.5) and (I.2.6)

Now suppose $u \in L_{p_1}(\Omega)$ for $1 \leqslant p_1 < \infty$, and the norms $\|u\|_{p_1}$ are uniformly bounded with respect to p_1 by some constant K. If $\|u\|_{\infty} > K$, or $u \notin L_{\infty}(\Omega)$, then we can find a constant $K_1 > K$, and a set $A \subset \Omega$ with $\mu(A) > 0$, such that $|u(x)| \geqslant K_1 \ \forall\, x \in A$. As before, it can be shown that

$$\lim_{p_1 \to \infty} \inf \|u\|_{p_1} \geqslant K_1 > K$$

which contradicts (I.2.2)