RING THEORY,

CONTINUOUS AND QUASI INJECTIVE MODULES

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CONTENTS

Summary

Quasi-injective modules were introduced by Johnson and Wong (10) as a generalisation of the concept of injective modules. In this work we study a generalisation of quasi-injective modules. We call a module M continuous if it satisfies the two conditions:

(i) Every submodule of M is essential in some direct summand of M, and

(ii) if a submodule A of M is isomorphic to a direct summand of M, then A is a direct summand.

The work in this thesis has been mainly influenced by the outstanding researches done by Faith, Johnson, Osofsky, Utumi and Wong.

In the first two chapters we state and prove some fundamental results of injective and quasi-injective modules. Some of these results are needed in the last chapter and some others are to be generalised.

The third chapter is devoted to the study of continuous modules. Several results on injective to hold (quasi-injective) modules are proved for continuous modules. It is shown by an example that a continuous module need not be quasi-injective. Among other results we prove that:

CHAPTER I

INJECTIVE MODULES

In this chapter we collect some fundamental properties of injective modules. The notions of free modules and projective modules are also discussed. As we were interested to point out only those results which will be used in the subsequent sections, we have ignored so many interested results on injective modules, some other results are stated without proof. The main results in this chapter are taken from [1], [2], [4], [5], [9], [11], [17].

1. PRELIMINARIES

All rings considered in this thesis have unity and all modules are untital right modules. If R is a ring, the notion mod-R will denote the set of all (right) R-modules. Let $M \in$ mod-R and $\{A_i : i \in I\}$ be a set of ~~all~~ submodules of M. Then

$$\left\{ \sum_{i \in I} a_i : a_i \in A_i,\ a_i = 0 \text{ for all but a finite number of } i\text{'s} \right\}$$

is the smallest submodule of M containing A_i for all i, this submodule is called the sum of $\{A_i : i \in I\}$ and is denoted by $\sum_{i \in I} A_i$. The family $\{A_i : i \in I\}$

is called <u>independent</u> in case

$$A_j \cap \sum_{j \neq i \in I} A_i = 0$$

for all $j \in I$. In this case we call $\sum_{i \in I} A_i$ a <u>direct sum</u> and denote it by $\bigoplus_{i \in I} A_i$.

Let $G = \{x_j : j \in J\}$ be a subset of M. Then $N = \sum_{j \in J} x_j R$, is called the submodule <u>generated</u> by G. G is said to be a set of <u>generators</u> for M in case M = N. Every module M has at least one set of generators, namely M. ($M = \sum_{x \in M} x R$). Let G be a set of generators of M. G is called a <u>basis</u> for M in case $\{x_j R : j \in J\}$ is independent. This is equivalent to saying that G is a set of generators and

$$\sum_{i \in I} x_i r_i = 0 \iff \text{every } r_i = 0,$$

A module M is said to be <u>finitely generated</u> if M has a finite set of generators.

An element $M \in$ mod-R is called <u>artinian</u> (<u>noetherian</u>) if every non-empty set of submodules of M has a minimal (maximal) element. This is the same as saying that every descending (ascending) sequence of submodules becomes ultimately stationary. It is known that a module is noetherian if and only if every submodule is finitely generated. If the module

R_R is artinian, then it is noetherian. However, the converse is not true. (Consider the ring of integers).

Let $M, N \in$ mod-R, and let $f : M \longrightarrow N$ be a mapping. f is called a homomorphism if for all $x, y \in M$ and all $r \in R$:

$$f(x + y) = f(x) + f(y)$$
$$f(x r) = f(x) r$$

Let M' and N' be subsets of M and N respectively. We define

$$f(M') = \{ f(x) : x \in M' \}$$
$$f^{-1}(N') = \{ y \in M : f(y) \in N' \}$$

$f(M')$ and $f^{-1}(N')$ are submodules of N and M respectively. $f(M)$ is called the image of f and is denoted by $\operatorname{Im} f$; $f^{-1}(o)$ is called the kernel of f and is denoted by $\operatorname{Ker} f$. f is called a monomorphism if $\ker f = o$, an epimorphism if $\operatorname{Im} f = N$. f is called isomorphism if it is a monomorphism and epimorphism; in this case we say that M is isomorphic to N and write $M \approx N$. If θ is a homomorphism of M into N; then

$$\operatorname{Im} \theta \approx M/\ker \theta .$$

Let $A, B, C \in$ mod-R. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be homomorphisms. We define

$gf : A \longrightarrow C$ by

$$g\,f\,(a) = g\,(f\,(a)\,)$$

for every $a \in A$. Clearly gf is a homomorphism. We note the following properties:

$$f \text{ and } g \text{ are mono.} \Longrightarrow g\,f \text{ is mono.}$$
$$f \text{ and } g \text{ are epi.} \Longrightarrow g\,f \text{ is epi.}$$
$$g\,f \text{ is mono.} \Longrightarrow f \text{ is mono.}$$
$$g\,f \text{ is epi.} \Longrightarrow g \text{ is epi.}$$

Let $\{ A_i : i \in I\}$ be a family of R-modules and let A denote their cartesian product. We may write any element of $a \in A$ in the form

$$a = [\ldots, a_i, \ldots] \quad (\text{or simply} = [a_i]).$$

Define operations on A componentwise, that is

$$a + b = [a_i + b_i] \text{ and } a\,r = [a_i\,r].$$

Then A becomes an R-module. We call A the <u>direct product</u> of $\{ A_i : i \in I\}$ and denote it by $\prod_{i \in I} A_i$. Define

$$p_j : A \longrightarrow A_j \text{ by } p_j\,(a) = a_j;$$
$$q_j : A_j \longrightarrow A \text{ by } q_j\,(x) = [a_i] \quad \text{where } a_i = \begin{cases} 0 & i \neq j \\ x & i = j \end{cases}$$

p_j is called the <u>natural projection</u> of A onto A_j

and q_j is called the natural injection of A_j into A. Then

$$p_k\, q_j = \begin{cases} 0 & k \neq j \\ 1_{A_j} & k = j \end{cases}$$

Let B be the subset of A consisting of all elements $a = [a_i]$ where $a_i \neq 0$ for only a finite subset of I. Then B is a submodule of A which is called the coproduct (or direct sum) of $\{A_i : i \in I\}$ and is denoted by $\bigoplus_{i \in I} M_i$. If $b \in B$, then $q_j\,(b_j) = [0]$ for almost all $j \in I$. Thus $\Sigma\, q_j\,(b_j)$ makes sense because Σ runs only over a finite number of terms.

Now

$$\Sigma\, q_j\, p_j\,(b) = \Sigma\, q_j\,(p_j\,(b)) = \Sigma\, q_j\,(b_j) = b$$

Hence

$$\Sigma\, q_j\, p_j = 1_B.$$

Let $M \in$ mod-R. Let $f_i : M \longrightarrow A_i$ be a family of homomorphisms. Then there exists a unique homomorphism

$$\phi : M \longrightarrow \prod_{i \in I} A_i$$

such that $p_j\, \phi = f_j$ for all $j \in I$

$$\begin{tikzcd} & M \arrow[dl, dashed, "\phi"'] \arrow[d, "f_j"] \\ \prod A_i \arrow[r, "p_j"'] & A_j \end{tikzcd}$$

If $h_i : A_i \longrightarrow M$ is a family of homomorphisms, then there exists a unique homomorphism

$$\theta : \bigoplus_{i \in I} A_i \longrightarrow M$$

$$\begin{tikzcd} & M \\ \bigoplus A_i \arrow[ur, dashed, "\theta"] & A_j \arrow[l, "q_j"] \arrow[u, "h_j"'] \end{tikzcd}$$

such that $\theta q_j = h_j$ for all $j \in I$

The above discussion shows that the notions of 'direct product' and 'direct sum' are dual to each other. If the index set I is finite, the two notions coincide.

An <u>ordered set</u> is a system $(S, \leq)$ where S is a set and $\leq$ is a binary relation which is reflexive, anti-symmetric and transitive. Let $A \subseteq S$, an element $x \in S$ is called an <u>upper bound</u> of A if $a \leq x$ for every $a \in A$; in this case we say that A is <u>bounded above</u>. A subset C of S is called a <u>chain</u> in S if for every $a, b \in C$ either $a \leq b$ or $b \leq a$. If every chain in S is bounded from above, we say that $(S, \leq)$ is an <u>inductive</u> set. An element $m \in S$ is called maximal if $m \nleq s$ for every $s \in S$. We will always have the occasion to use the fundamental axiom known as Zorn's Lemma which states that every inductive set has at least one maximal element.

2. EXACT SEQUENCES.

2.1. DEFINITIONS. Let $\{ M_i : i \in I \}$ be a countable set of modules with corresponding collection of mappings $f_i : M_i \longrightarrow M_{i+1}$. Then the sequence

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \cdots$$

is called exact provided that $\operatorname{Im} f_{i-1} = \ker f_i$ for every $i \in I$. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a short exact sequence

Using the notion of **exact** sequences, equivalent definitions of epimorphisms, monomorphisms and isomorphisms may be given as follows: A homomorphism $f : M \longrightarrow N$ is an epimorphism, monomorphism or isomorphism according as:

$$M \xrightarrow{f} N \longrightarrow 0, \quad 0 \longrightarrow M \xrightarrow{f} N \text{ or } 0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$$

is exact, respectively.

An exact sequence $M \xrightarrow{f} N \longrightarrow 0$ ($0 \longrightarrow M \xrightarrow{f} N$) is said to split if there is a homomorphism $g : N \longrightarrow M$ such that $f\, g = 1_N$ ($g\, f = 1_M$), the homomorphism g is called the splitting homomorphism. An exact sequence that splits is called split exact.

2.2. **PROPOSITION.** Let $M \xrightarrow{j} N \longrightarrow o$ be split exact with splitting homomorphism k. Then

$$M = I\,m\,k \oplus \ker j \approx N \oplus \ker j \approx I\,m\,j \oplus \ker j.$$

Proof. Note that Im k and ker j are submodules of M and

$$I\,m\,k \cap \ker j = 0.$$

Let m be any element in M, then

$$j\,(m - k\,j\,(m)) = j\,(m) - (j\,k)\,j\,(m) = j\,(m) - j\,(m) = o.$$

Thus $m - k\,j\,(m) \in \ker j$, whence $m \in \operatorname{Im} k \oplus \ker j$. Therefore

$$M = \operatorname{Im} k \oplus \operatorname{Ker} j.$$

Since j is an epimorphism and k is a monomorphism,

$$\operatorname{Im} k \approx N = \operatorname{Im} j.$$

This completes the proof.

Similar to the proof of the above Proposition, one can easily prove the following:

2.3. **PROPOSITION.** Let $0 \longrightarrow N \xrightarrow{j} M$ be a split exact sequence with splitting homomorphism k. Then

$M = \mathrm{Im}\ j \oplus \mathrm{Ker}\ k \approx N \oplus \mathrm{Ker}\ k \approx \mathrm{Im}\ k \oplus \mathrm{Ker}\ k.$

Next we prove:

2.4. **THEOREM.** If a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

splits at one end, then it splits at the other end and $B \approx A \oplus C$.

Proof. Suppose that $B \xrightarrow{g} C \longrightarrow 0$ splits with splitting homomorphism g'. By proposition 2.2.

$$B = \mathrm{Im}\ g' \oplus \mathrm{Ker}\ g.$$

since $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$, is exact
$\mathrm{Ker}\ g = \mathrm{Im}\ f.$ So that

$$B = \mathrm{Im}\ g' \oplus \mathrm{Im}\ f$$

As f and g' are monomorphisms

$$B \approx A \oplus C.$$

Let h be the projection of B onto $\mathrm{Im}\ f$. Define
$f' : B \longrightarrow A$ by $f'(b) = f^{-1}(h(b))$. Then
$f' f = 1_A$. This completes the proof.

3. PROJECTIVE MODULES.

We start by defining and discussing a more restricted class of modules.

3.1. DEFINITION. A module M is called free if M has a basis (we may consider the empty set as a basis for the zero module).

It follows immediately by the definition that any ring R is free. (In fact $\{1\}$ is a basis for the ring R). The following theorem generalizes this observation.

3.2. THEOREM. An element M in mod-R is free if and only if M is isomorphic to a direct sum of copies of R.

Proof. Let $\{m_i : i \in I\}$ be a basis of M. For each $i \in I$, $m_i R \approx R$. Now

$$M = \sum_{i \in I} m_i R = \bigoplus_{i \in I} m_i R,$$

whence the 'only if' part follows. Conversely, suppose that $M \approx \bigoplus_{i \in I} R_i$, where each $R_i \approx R$ as a right R-modules. Let f be the given isomorphism of $\bigoplus_{i \in I} R_i$ onto M and g_i the isomorphism of R onto R_i. Let

$$m_i = f g_i (1) .$$

Then it is clear that $\{m_i : i \in I\}$ is a basis of M.