SPECIAL FUNCTIONS AND INTEGRAL TRANSFORMS

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TO

MY PARENTS

&

MY HUSBAND

بسم الله الرحمن الرحيم

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The theory of generalized hypergeometric function is fundamental in the field of mathematical physics, since all the commonly used functions of analysis "Bessel Functions, Legendre Functions, etc." are special cases of the general functions. The unified theory provides a means for the analysis of the simpler functions and can be used to solve the more complicated equations in physics.

The subject of this thesis is the higher transcendental function known as the confluent hypergeometric function. In the last two decades this function has taken an ever increasing significance because of its use in the application of mathematics to physical and technical problems. The designation of the function under discussion as the confluent hypergeometric function is unique in so far as it is derived from the hypergeometric function of Gauss $_2F_1(a,b;c;z)$.

This thesis is divided into eight chapters. The first is devoted to some basic formulas such as the definitions of the generalized hypergeometric series and Kummer's function, the differential formula to Kummer's function also the integral formula to it, the integral representation for Kummer's function, some functions which are related to ${}_1F_1\ (a;c;z)$ and finally we devoted to the form of Kummer's differential equation.

The second chapter consists of four sections which denote linear relations between some special cases of ${}_1F_1\ (a;c;z)$.

Some new relations will be introduced such as :

(i) $(c-1)\ {}_1F_1\ (a-1;\ c-1;z)-a\ {}_1F_1\ (a+1;c;z)\ +(1+a-c+z)\ {}_1F_1\ (a;c;z) = 0.$

(ii) $z(a-1)\ {}_1F_1\ (c+1;c-a+3;z)+(a-c-1)(a-c-2)\ {}_1F_1(c+1;c-a+2;z) = (a-c-1)\ (a-c-2)\ {}_1F_1\ (c;c-a+1;z).$

Chapter three deals with Mac - Robert E-function and its recurrence relations. Section one of this chapter will be concerned with the definition of the E-function while in section two, we prove that the E-function satisfies the differential equation,

$$z^2.\ w''+\ (1-a-b-z)\ z.\ w'\ +\ ab.\ w = 0.$$

We shall be concerned in chapter four with porducts of Kummer's functions. In section one of this chapter, we shall obtain the following new theorem,

$$ {}_1F_1(a;c;z)\,{}_1F_1(-a;1-c;-z) - \frac{az}{c(1-c)}\,{}_1F_1(a+1;c+1;z)\,{}_1F_1(1-a;2-c;-z) = 1 $$

Our proof is based on the expansion of each ${}_1F_1$ by its power series, then rearranging the repeated series. Also we shall give an alternating proof for a particular case of the last theorem which is based on the application of the differential equation satisfied by ${}_1F_1$. At the end of this chapter, an infinite series of product of two ${}_1F_1$ is given in terms of one ${}_2F_3$.

Chapter five deals with multiplication and addition theorems. Some new theorems such as,

$$ \text{(i)}\quad e^{z-\lambda}\left(\frac{\lambda-z}{\lambda}\right)^{a-c} {}_1F_1\left(\begin{matrix} c-a;\, z+\lambda \\ c \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(c-a)_n}{n!} \cdot {}_1F_1\left(\begin{matrix} a-n;\, z-\lambda \\ c \end{matrix}\right) \cdot \left(\frac{2z}{z+\lambda}\right)^n $$

$$ \text{(ii)}\quad {}_1F_1\left(\begin{matrix} a;\, z \\ c \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!\,(c)_n} \cdot {}_1F_1\left(\begin{matrix} a+n ;\, \dfrac{z(\lambda+1)}{\lambda} \\ c+n \end{matrix}\right) \cdot \left(\frac{-z}{\lambda}\right)^n, $$

will be introduced.

Chapter six involves integrals for the confluent hyper-geometric function. In section one, we evaluate some integrals involving Bessel's functions while in sections two and three two integrals will be concerned involving ${}_1F_1$ and ${}_1F_2$ respectively. Sections four and seven are devoted to some intagrals involving the exponential function, trigonometric functions, and ${}_1F_1$ with Bessel's function respectively. Integrals of products of two ${}_1F_1$ are established in the rest of this chapter.

In chapter seven, we evaluate an integral involving ${}_1F_1$ in terms of the generalized hypergeometric function. We also evaluate integrals involving the product of ${}_1F_1$ and the exponential fúnction. Finally integrals of products of two ${}_1F_1$ are evaluated in terms of the generalized hypergeometric functions and in terms of product of two ${}_2F_1$.

Chapter eight involves some integral transforms. In particular we shall be concerned with fourier, Laplace, Mellin, Hankel, the KJ, the J_ν, Y_ν and finally the $\left[J^2_{-\nu}, -J^2_\nu\right]$ transforms.

S. Sheh
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CHAPTER I

BASIC FORMULAS

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BASIC FORMULAS

1. Definition of the Generalized Hypergeometric Series:

The generalized hypergeometric function is defined by the following:

$$ {}_pF_q \left(\begin{matrix} a_1, a_2, \ldots, a_p; z \\ b_1, b_2, \ldots, b_q \end{matrix} \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{n! (b_1)_n (b_2)_n \cdots (b_q)_n} z^n . \quad (1.1) $$

where,

$$ (a)_n = a(a+1)(a+2)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad (1.2) $$

$$ (a)_0 = 1 . $$

The values of the q parameters $b_1, b_2, \ldots, b_q$ are always different from $0, -1, -2, \ldots$. In this notation we get,

$$ F(a,b;c;z) = {}_2F_1(a,b;c;z) . $$

${}_pF_q$ converges for all finite z if $p \leq q$, converges for $|z| < 1$ if $p = q+1$, and diverges for all $z \neq 0$ if $p > q+1$. This excludes certain integer values of the parameters for which the series terminates or fails to make sense. If one or more of the parameters $a_1, a_2, \ldots, a_p$ is a negative integer, the series terminates; the cases in which one of the parameters $b_1, b_2, \ldots, b_q$ is a negative integer are excluded.

2. Kummer's Function:

Kummer's function ${}_1F_1\,(a;c;z)$ is defined by,

$$ {}_1F_1\,(a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!\,(c)_n} z^n . \tag{2.1}$$

$c \neq 0, -1, -2, \ldots$

This series converges for all z provided that $c \neq -m$, where $m = 0, 1, 2, \ldots$.

The following are the cases of the convergence of the series:

(i) If $c \neq -m$, $m = 0, 1, 2, \ldots$,
then the series converges for all z.

(ii) If $c \neq -m$, $a=-n$, m, n non-negative integers,
then the series reduces to a polynomial of degree n in z.

(iii) If $a \neq -n$, $c= -m$ or $a = -n$, $c=-m$ and $m<n$
where m, n are non-negative integers,
then the series does not converge.

(iv) Although the function ${}_1F_1\,(a;c;z)$ is undefined if $c=-m$, $m= 0, -1, -2, \ldots$, the limit of $\frac{1}{\Gamma(c)}\,{}_1F_1\,(a;c;z)$ as $c \longrightarrow -m$ exists and is given by,

$$\lim_{c \to -m} \frac{1}{\Gamma(c)}\,{}_1F_1\,(a;c;z) = \frac{z^{m+1}}{(m+1)!}(a)_{m+1}\;{}_1F_1(a+m+1;m+2;z).$$

3. Principal Formulas :

Some useful results are,

$$\frac{d}{dz}\ {}_1F_1(a;c;z) = \frac{a}{c}\ {}_1F_1(a+1;c+1;z). \tag{3.1}$$

$$\frac{d^n}{dz^n}\ {}_1F_1(a;c;z) = \frac{(a)_n}{(c)_n}\ {}_1F_1(a+n;c+n;z). \tag{3.2}$$

$$\int_0^z {}_1F_1(a;c;z)\,dz = \frac{(c-1)}{(a-1)}\ {}_1F_1(a-1;c-1;z). \tag{3.3}$$

$$(a \neq 1)$$

$$\int_0^\infty t^{c'-1}\ {}_1F_1(a;c;-t)\,dt = \frac{\Gamma(c)\,\Gamma(c')\,\Gamma(a-c')}{\Gamma(a)\,\Gamma(c-c')} \tag{3.4}$$

(See [1])

$$(\mathrm{Re}\ a > \mathrm{Re}\ c' > 0).$$

4. Integral Representation for Kummer's Function :

(See [2] P. 39-63 and P. 127-172)

The formula,

$$B(a,c-a)\ {}_1F_1(a;c;z) = \int_0^1 e^{zt}\, t^{a-1}(1-t)^{c-a-1}\,dt. \tag{4.1}$$

holds if Re $a > 0$, Re $(c-a) > 0$. It can be verified by expanding the exponential function in powers of (zt) and then integrating term by term. If $(1-t)$ is written for t the integral becomes,