

**MAGNETOHYDRODYNAMIC
DISTURBANCES IN A VISCOUS FLUID
OF FINITE ELECTRICAL CONDUCTIVITY
(Symmetric Case)**

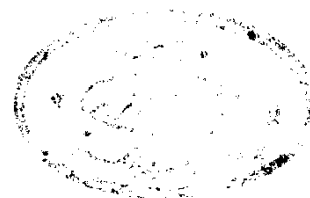
A Thesis Submitted in Partial Fulfilment of
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Master of Science Degree

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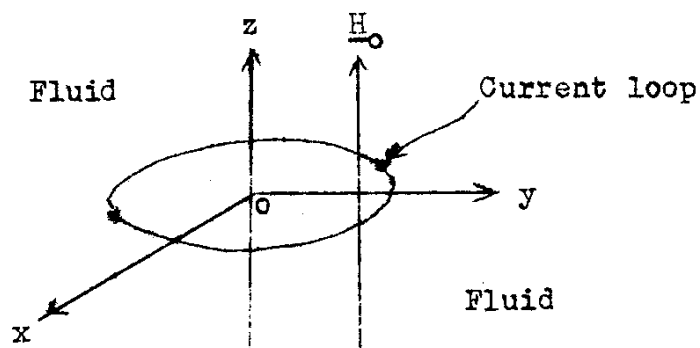
SUMMARY

It is supposed that we have initially an INFINITE mass at rest of a VISCOUS fluid of finite electrical conductivity in which a uniform field $\underline{H}_0$ is prevailing. It is supposed then that a dipole of moment $\underline{M}$ is suddenly introduced in the above configuration of the system. As a result of this sudden introduction a velocity $\underline{U}$ is produced in a certain volume of the fluid as well as a small disturbance $\underline{h}$ of $\underline{H}_0$. The dipole, whose axis is always taken in the z-direction, can be replaced by a small current loop in the (x,y)-plane, of strength I, radius a, with its centre always fixed at the origin of coordinates, and such that

$$\underline{M} = \lim_{a \rightarrow 0} \pi a^2 \underline{I} \quad \dots \quad \dots \quad \dots \quad (1)$$

$$\underline{I} = \underline{1}_\varphi I \delta(\rho - a) \cdot H(t) \quad \dots \quad \dots \quad \dots \quad (2)$$

where $\underline{1}_\varphi$ is a unit vector in the direction of φ increasing, (ρ, φ, z) are cylindrical polar coordinates, $H(t)$ is the Heaviside unit function, and δ is Dirac's delta function. Only the SYMMETRIC case, where the uniform field $\underline{H}_0$ is always taken parallel to the z-axis, parallel to the dipole axis, is dealt with.



SYMMETRIC CASE

Uniform field $\underline{H}_0 = H_0 \underline{1}_z$ parallel to the dipole axis

In chapter I of this work boundary conditions on crossing the loop have been introduced in the basic equations used, namely Maxwell's equations and the Eulerian hydrodynamic equation.

In chapter II, mathematical solutions representing the behaviour of both $\underline{U}$ and $\underline{h}$ have been re-studied in the case where a MAGNETIC dipole has been suddenly introduced. Extensive figures calculated from the obtained mathematical results have been drawn. The behaviour of the resulting magnetohydrodynamic disturbances $\underline{h}$ as well as of fluid motion $\underline{U}$ have been traced of different intervals of time.

It is worth to note here that the figures so obtained, in the case of a VISCOUS fluid, show that the effect of

viscosity in the case of an infinite fluid, is not of great importance. In fact, it is found that there is a great similarity between the behaviour of both $\underline{U}$ and $\underline{h}$ in this case and those obtained by MOHANDIS (1959) in a nonviscous fluid.

In chapter III the same problem has been solved for a viscous fluid, but with the sudden introduction of an OSCILLATING dipole instead of a magnetic one. This case has been already published as a joint work in ILNUOVO CIMENTO (1969), Series X, Vol. 59B, pp. 1-11. It is seen that the solution of the problem of an oscillating dipole is much more difficult than that of a magnetic one. A new tedious effort is needed to represent the obtained mathematical results in a form suitable for computation.

CHAPTER I

DIPOLE IN A VISCOUS FLUID

§ I.1- FUNDAMENTAL EQUATIONS

The phenomena can be described by using Maxwell's equations and the Eulerian hydrodynamic equation. Maxwell's equations in this case and in the presence of a moving electrically conducting matter with velocity $\underline{U}$ and electrical conductivity σ are :

$$\text{Curl } \underline{E} = - \frac{1}{c} \frac{\partial \underline{H}}{\partial t} \quad \dots \quad \dots \quad \dots \quad (3)$$

$$\text{Curl } \underline{H} = \frac{4\pi}{c} \underline{J} \quad \dots \quad \dots \quad \dots \quad (4)$$

$$\text{where} \quad \underline{J} = \sigma (\underline{E} + \underline{U} \wedge \underline{H}) \quad \dots \quad \dots \quad (5)$$

$$\text{also} \quad \text{div } \underline{H} = 0 \quad \dots \quad \dots \quad \dots \quad (6)$$

$$\text{div } \underline{J} = 0 \quad \dots \quad \dots \quad \dots \quad (7)$$

where $\underline{E}$ is the electric field, $\underline{J}$ the electric current and $\underline{H}$ is the magnetic field.

Making the substitution

$$\underline{H} = \underline{H}_0 + \underline{h} \quad \dots \quad \dots \quad \dots \quad (8)$$

$$\text{where} \quad \underline{h} \ll \underline{H}_0 \quad \dots \quad \dots \quad \dots \quad (9)$$

$$\text{div } \underline{h} = 0 \quad \dots \quad \dots \quad \dots \quad (10)$$

$$\underline{H}_0 = H_0 \underline{1}_z \quad \dots \quad \dots \quad \dots \quad (11)$$

where $\underline{1}_z$ is a unit vector in the direction of z , taking the curl of equations (4) and (5), substituting in equation (3), applying conditions (6) and (7), we see on neglecting squares and products of $\underline{U}$ and $\underline{h}$, that

$$\left(\frac{\partial}{\partial t} - \frac{1}{4\pi\sigma} \nabla^2\right)\underline{h} = (\underline{H}_0 \cdot \text{grad}) \underline{U} \quad \dots \quad \dots \quad (12)$$

The Eulerian hydrodynamic equation in the case of an incompressible viscous fluid is

$$\rho \frac{d \underline{U}}{dt} = \underline{F} - \nabla p + \rho \gamma \nabla^2 \underline{U} \quad \dots \quad \dots \quad (13)$$

where the moving matter is considered to be of density ρ , hydrostatic pressure p , and kinematic viscosity γ .

Also the equation of continuity shows that

$$\text{div } \underline{U} = 0, \quad \dots \quad \dots \quad \dots \quad (14)$$

The pondermotive force acting on the loop is

$$\underline{F} = \underline{J} \wedge \underline{H} \quad \dots \quad \dots \quad \dots \quad (15)$$

Substituting from equations(8) and (15) in equation (13), we see on neglecting squares and products of $\underline{U}$ (the fluid velocity), and of $\underline{h}$ (the magnetic perturbation), that

$$\frac{\partial \underline{U}}{\partial t} = \frac{H_0}{4\pi\rho} \frac{\partial \underline{h}}{\partial z} - \text{grad } \frac{1}{\rho} \left[p + \frac{1}{8\pi} (\underline{H}_0 + \underline{h})^2 \right] + \gamma \nabla^2 \underline{U} \quad (16)$$

Putting

$$\underline{P} = - \text{grad } \tilde{\omega} , \quad \dots \quad \dots \quad \dots \quad (17)$$

where

$$\tilde{\omega} = \frac{1}{\rho} \left[p + \frac{1}{8\pi} (\underline{H}_0 + \underline{h})^2 \right] , \quad \dots \quad (18)$$

taking the divergence of equation (16) and using conditions (10) and (14), we find that

$$\text{div } \underline{P} = 0. \quad \dots \quad \dots \quad \dots \quad (19)$$

It is found more convenient to express equations (12) and (16) in non-dimensional units. This can be done by introducing new variables defined by

$$\left. \begin{aligned} \underline{h} &= H_0 \underline{h}' , \quad \underline{U} = V \underline{U}' , \quad \underline{P} = \omega_0 V \underline{P}' , \\ t' &= \omega_0 t , \quad \underline{r} = \omega_0 r' / V , \quad \gamma = \gamma' V^2 / \omega_0 , \end{aligned} \right\} \quad (20)$$

where V is the Alfvén velocity given by

$$V^2 = H_0^2 / 4 \pi \rho \quad \dots \quad \dots \quad \dots \quad (21)$$

and ω_0 is a frequency defined by

$$\omega_0 = \sigma H_0^2 / \rho \quad \dots \quad \dots \quad \dots \quad (22)$$

Henceforth the primes will be omitted.

Taking z as a co-ordinate measured in the direction of $\underline{H}_0$, the basic equations (12) and (16), in terms of

these nondimensional quantities become

$$(\partial / \partial t - \nabla^2) \underline{h} = \partial \underline{U} / \partial z, \quad \dots \quad (23)$$

$$\partial \underline{U} / \partial t = \partial \underline{h} / \partial z + \underline{P} + \gamma \nabla^2 \underline{U} \quad \dots \quad (24)$$

Introducing the solenoidal vectors $\underline{\psi}$, $\underline{x}$ and $\underline{\xi}$ such that

$$\underline{h} = \text{curl } \underline{\psi}, \quad \underline{U} = \text{curl } \underline{x} \quad \dots \quad (25)$$

and

$$\underline{P} = \text{curl } \underline{\xi} \quad \dots \quad (26)$$

equations (23) and (24) reduce respectively to the form

$$(\partial / \partial t - \nabla^2) \underline{\psi} = \partial \underline{x} / \partial z, \quad \dots \quad (27)$$

$$(\partial / \partial t - \gamma \nabla^2) \underline{x} = \partial \underline{\psi} / \partial z + \underline{\xi}. \quad \dots \quad (28)$$

Since $\underline{P}$ has been expressed in equation (17) as the gradient of a scalar quantity, it follows from equation (26) that

$$\nabla^2 \underline{\xi} = 0. \quad \dots \quad (29)$$

§ 1.2- INTEGRAL TRANSFORMS USED

1. The Heaviside transform denoted by broken brackets around the transformed symbol. It is defined by

$$\langle a(\underline{r}, p) \rangle = p \int_0^\infty a(\underline{r}, t) \exp(-pt) dt \quad (30)$$

Integration by parts shows that

$$\langle \partial a / \partial t \rangle = p(\langle a \rangle - a_0) ,$$

$$\langle \partial^2 a / \partial t^2 \rangle = p^2(\langle a \rangle - a_0) - p a_1 ,$$

where

$$a_0 = a(r,0), \quad a_1 = \partial a(r,0) / \partial t ,$$

a_0 and a_1 will be neglected in our results, since initially the field is uniform and the liquid is at rest.

2. The Hankel transform denoted by a superscript star, thus :

$$a^*(\xi, z, t) = \int_0^\infty a(\rho, z, t) J_1(\xi \rho) \rho d\rho \quad (31)$$

the inversion to which is

$$a(\rho, z, t) = \int_0^\infty a^*(\xi, z, t) J_1(\xi \rho) \xi d\xi \quad (32)$$

3. The Fourier transform denoted by a superscript dagger, thus;

$$a^+(\xi, n, t) = \int_{-\infty}^\infty a(\xi, z, t) e^{inz} dz. \quad (33)$$

It follows from the Fourier integral theorem that

$$a(\xi, z, t) = \frac{1}{2\pi} \int_{-\infty}^\infty a^+(\xi, n, t) e^{-inz} dn.$$

4. The double Fourier transform denoted by a superscript star, thus

$$\psi^*(\ell, m, z, t) = \iint_{-\infty}^\infty \psi(x, y, z, t) \exp[i(\ell x + m y)] dx dy \quad (34)$$

the solution to which is

$$\psi(x, y, z, t) = \frac{1}{4\pi^2} \iint_{-\infty}^\infty \psi^*(\ell, m, z, t) \exp[-i(\ell x + m y)] d\ell dm$$

§ 1.3- TO DEFINE $\underline{I}^{\#}$

Let I_x and I_y be the x and y components of $\underline{I}$ as defined in the summary by

$$\underline{I} = \underline{I}_0 \delta(\rho - a) \cdot H(t) ;$$

then

$$I_x = - \underline{I} \sin \theta , \quad I_y = \underline{I} \cos \theta \quad \dots \quad \dots \quad (35)$$

Let $\ell = \xi \cos \alpha$, $m = \xi \sin \alpha$

then

$$\xi^2 = \ell^2 + m^2 \quad \dots \quad \dots \quad \dots \quad (36)$$

and $\ell x + m y = \rho \xi \cos (\theta - \alpha)$

also

$$I_x^{\#} = -i \underline{I}^{\#} \sin \alpha , \quad I_y^{\#} = i \underline{I}^{\#} \cos \alpha \quad \dots \quad (37)$$

It then follows that

$$\underline{I}^{\#} = i I_x^{\#} \sin \alpha - i I_y^{\#} \cos \alpha$$

Applying the double Fourier transform denoted by equation (34) to equations (35) and substituting in equation (37), to obtain values for $I_x^{\#}$ and $I_y^{\#}$, we have

$$\begin{aligned} I_x^{\#} &= - I \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin \theta \delta(\rho - a) e^{i(\ell x + m y)} dx dy \\ &= - i a I \sin \alpha J_1(a \xi) \end{aligned}$$

also

$$I_y^{\#} = ia I \cos \alpha J_1(a \xi)$$

and

$$\underline{I}^{\#} = \xi \underline{M} \quad \dots \quad \dots \quad \dots \quad (38)$$

§ I.4- THE BOUNDARY CONDITIONS

On crossing the plane $z = 0$, certain boundary conditions have to be satisfied at the loop, these can be derived as follows. Since

$$\underline{h} = \text{curl} (0, \underline{\psi}(\rho, z, t), 0)$$

and since

$$(\underline{1}_z \wedge \underline{h})_+ - (\underline{1}_z \wedge \underline{h})_- = 4\pi I \underline{1}_\phi \delta(\rho - a),$$

therefore

$$\underline{\psi}_+ - \underline{\psi}_- = 0 \quad \dots \quad \dots \quad \dots \quad (39)$$

and

$$\left(\frac{\partial \underline{\psi}}{\partial z} \right)_+ - \left(\frac{\partial \underline{\psi}}{\partial z} \right)_- = -4\pi I \underline{1}_\phi \delta(\rho - a) \quad (40)$$

Since

$$\underline{U} = \text{curl} (0, \underline{\chi}(\rho, z, t), 0),$$

therefore

$$\underline{\chi}_+ - \underline{\chi}_- = 0 \quad \dots \quad \dots \quad \dots \quad (41)$$

and

$$\left(\frac{\partial \underline{\chi}}{\partial z} \right)_+ - \left(\frac{\partial \underline{\chi}}{\partial z} \right)_- = 0 \quad (42)$$