SYSTEMS OF INTEGRAL EQUATIONS OF FREDHOLM TYPE

A THESIS

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CONTENTS

SUMMARY

The aim of the present thesis is :

(I) The study of the theory of existence and uniqueness of solution of a system of integral equations of Fredholm type.

(II) The study of the spectral properties of a system of integral equations in the case of a symmetric matrix-kernel.

The thesis consists of three chapters :

The first chapter serves as an introduction, it contains some basic concepts from functional analysis, these concepts are used in chapter II and III.

In §1.1 we have formulated the problem of the thesis.

In §1.2 we have studied some analytic properties of $C^{(n)}(a,b)$, like completness of $C^{(n)}(a,b)$, and compactness of its subsets.

§1.3 deals with the same problems for the space $L_2^{(n)}(a,b)$.

In the second chapter attention is focused on studing the existence and uniqueness of solution of the problem

$$\Phi(x) - \lambda \int_a^b K(x,y)\, \Phi(y)dy = F(x) \qquad (I)$$

We have used the last theorem to find a solution of the problem when the kernel K(x,y) is a symmetric matrix-kernel, this solution is represented by the absolutely and uniformly convergent series.

$$\Phi(x) = F(x) + \lambda \sum_{h=1}^{\infty} \frac{A_h}{\lambda_h - \lambda} \Phi_h \qquad \lambda_h \neq \lambda$$

$$(h = 1, 2, \ldots)$$

Also we have proved that the series $\sum_{h=1}^{\infty} \frac{\Phi_h(x) \otimes \Phi_h(y)}{\lambda_h^m}$

is convergent in the mean to the iterated matrix-kernel $K_m(x,y)$ where

$$K_m(x,y) = \int_a^b K(x,y)\, K_{m-1}(y,z)\,dy.$$

CHAPTER I

BASIC CONCEPTS FROM ANALYSIS

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§1.1. Introduction and notations :

Our aim is the study of a system of Fredholm's integral equations from the point of view of the existence and uniqueness of solution and its spectral properties; i.e. We generalise the known results concerning one Fredholm's integral equation to the case when the problem deals with a system of Fredholm's integral equations.

The study of one Fredholm's integral equation is deeply considered in many references [1],[2] , [9].

Our system will be written in the form

$$a_{11}\varphi_1(x) + a_{12}\varphi_2(x) + \dots + a_{1n}\varphi_n(x) - \lambda\int_a^b \left[k_{11}(x,y)\varphi_1(y) + \dots + k_{1n}(x,y)\varphi_n(y)\right]dy = f_1(x)$$

$$a_{21}\varphi_1(x) + a_{22}\varphi_2(x) + \dots + a_{2n}\varphi_n(x) - \lambda\int_a^b \left[k_{21}(x,y)\varphi_1(y) + \dots + k_{2n}(x,y)\varphi_n(y)\right]dy = f_2(x)$$

$$\vdots$$

$$a_{n1}\varphi_1(x) + a_{n2}\varphi_2(x) + \dots + a_{nn}\varphi_n(x) - \lambda\int_a^b \left[k_{n1}(x,y)\varphi_1(y) + \dots + k_{nn}(x,y)\varphi_n(y)\right]dy = f_n(x)$$

Using the matrix notation, this system can be simply written in the form

$$A\,\Phi(x) - \lambda \int_a^b K(x,y)\,\Phi(y)\,dy = F(x) \qquad (1.1.1)$$

where;

$$A = \begin{bmatrix} a_{11} a_{12} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} a_{n2} & \cdots & a_{nn} \end{bmatrix} , \quad K(x,y) = \begin{bmatrix} k_{11}(x,y) & \cdots & k_{1n}(x,y) \\ \vdots & & \vdots \\ k_{n1}(x,y) & \cdots & k_{nn}(x,y) \end{bmatrix}$$

$$F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} , \qquad \Phi(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_n(x) \end{bmatrix}$$

($K(x,y)$ is called the matrix-kernel of the system).

In the above notations k_{ij} $(i,j = 1,2,\ldots n)$ are given functions of (x,y), λ is a parameter, $f_i(x)$ $(i = 1,2,\ldots n)$ are given functions and $\varphi_i(x)$ are the unknown functions.

The above system of integral equations are known as a system of Fredholm's integral equations, they are so called in honour of Fredholm who deeply studied one integral equation of such type [1], [2] , [9].

If the vector-function $F(x) \neq o$; the system of linear Fredholm's integral equations of the second kind is called non-homogeneous, in other cases (i.e. when $F(x) = o$) the system of Fredholm's integral equations is called homogeneous.

If $\lambda = o$ then the system becomes

$$\int_a^b K(x,y)\, \Phi(y)\, dy = F(x)$$

which is called a system of linear Fredholm's integral equations of the first kind.

It is reasonable to assume that each component of the vector

$$F(x) = \begin{bmatrix} f_1(x) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x) \end{bmatrix}$$ belongs to some functional space, for example

$C(a,b)$ (which consists of all continuous functions $f(x)$ defined in the closed interval $[a,b]$) consequently, the vector F will belong to the cartesian product $\underbrace{C(a,b) \times C(a,b) \times \ldots \times C(a,b)}_{n\text{-times}}$ which will be denoted by $C^n(a,b)$, if each component in $K(x,y)$ belongs to the space $C(a,b)^2$ hence, the matrix $K(x,y)$ will belong to the cartesian product

$$\underbrace{C(a,b)^2 \times C(a,b)^2 \times \ldots \times C(a,b)^2}_{n^2\text{-times}}$$

which will be denoted $C^{(n^2)}(a,b)^2$

On the other hand, we may assume that each component of the vector $F(x) = \begin{bmatrix} f_1(x) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x) \end{bmatrix}$ belongs to the space $L_2(a,b)$ (i.e. the space of all measurable functions such that $\int_a^b |f(x)|^2 dx < \infty$), so we introduce the space

$$L_2^{(n)}(a,b) = \underbrace{L_2(a,b) \times \ldots \times L_2(a,b)}_{n\text{-times}}$$

consequentely, $F(x) \in L_2^{(n)}(a,b)$. If each component in $K(x,y)$ belongs to the space $L_2(a,b)^2$, then $K(x,y)$ belongs to the space $L_2^{(n^2)}(a,b)^2$ which is defined as the cartesian product :-

$$\underbrace{L_2(a,b)^2 \times L_2(a,b)^2 \times \ldots \times L_2(a,b)^2}_{n^2\text{-times}}$$

If the inverse A^{-1} of the matrix A exists then by multiplying system (1.1.1) by A^{-1}, it takes the form

$$\Phi(x) - \lambda \int_a^b A^{-1}K(x,y)\,\Phi(y)dy = A^{-1}F(x) \qquad \text{or}$$

$$\Phi(x) - \lambda \int_a^b K_1(x,y)\,\Phi(y)\,dy = F_1(x) \qquad (1.1.2)$$

with $K_1(x,y) = A^{-1}K(x,y)$ and $F_1(x) = A^{-1}F(x)$

We shall concentrate our study on systems of the form (1.1.2), our study will take into account three types of the matrix-kernel K(x,y).

Firstly :- the kernel is sufficiently small in norm (according to the space to which it belongs), in this case we shall prove the existence and uniqueness of solution of the system, the method of proof is based on the principle of contraction mappings which is sometimes called the method of successive approximations.

Secondly :- each component in K(x,y) is degenerate; i.e.

$$k_{ij}(x,y) = \sum_{\ell=1}^{n_{ij}} a_{\ell}^{(ij)}(x)\, b_{\ell}^{(ij)}(y) \qquad \text{where}$$

$a_1^{(ij)}(x)$, $a_2^{(ij)}(x)$, ... $a_{n_{ij}}^{(ij)}(x)$, $b_1^{(ij)}(y)$, $b_2^{(ij)}(y)$,... $b_{n_{ij}}^{(ij)}(y)$ are two families of linearly independent functions, in this case we shall prove the existence and uniqueness of solution of the system (1.1.2) for all values of the parameter λ except for a finite number of values.

Thirdly :- each component $k_{ij}(x,y)$ of the matrix-kernel K(x,y) defines a complete continuous integral operator, in this case the matrix K(x,y) is decomposed into two parts one is of small norm and the other is degenerate, a combination of the first and second cases helps us in the study of this third type.

Now we give a brief analytic account of the two spaces $C^n(a,b)$, and $L_2^n(a,b)$.

§1.2. Analytic properties of $C^n(a,b)$:-

The space $C^n(a,b)$ consisting of all vector-functions $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ is a linear space under usual operations of addition and multiplication by a scalar :

If $F(x) = (f_1(x), f_2(x), \ldots f_n(x))$,

$G(x) = (g_1(x), g_2(x), \ldots g_n(x))$ then

$$F(x) + G(x) = (f_1(x) + g_1(x), f_2(x) + g_2(x), \ldots, f_n(x)+g_n(x))$$

and for any $\alpha \in \mathbb{C}$

$$\alpha F(x) = (\alpha f_1(x), \alpha f_2(x), \ldots, \alpha f_n(x)).$$

By introducing the following norm

$$\|F\|_{C^n} = \max \left\{ \sup_{x \in (a,b)} f_1(x) , \sup_{x \in (a,b)} f_2(x) , \ldots, \sup_{x \in (a,b)} f_n(x) \right\}$$

$C^n(a,b)$ becomes a normed space.

Completeness of $C^n(a,b)$:-

In general metric spaces, the metric space X is called complete if, and only if, each fundamental sequence of X has a limit in X [4] , [5]

Theorem 1.2.1.

The space $C^n(a,b)$ is complete; i.e. each fundamental sequence in $C^n(a,b)$ has a limit in $C^n(a,b)$.

Proof :

Let $F_m(x) = \left\{ f_1^{(m)}(x), f_2^{(m)}(x), \ldots, f_n^{(m)}(x) \right\}$, $m = 1,2,\ldots$ be a fundamental sequence in $C^n(a,b)$ i.e.

for all $\varepsilon > o$ there exists $N > \mathbb{N}$ such that for all $\ell > N$ & $m > N$

, $\| F_\ell(x) - F_m(x) \|_{C^n(a,b)} < \varepsilon$ this, in turn, means that

for all $\varepsilon > o$ there exists $N \in \mathbb{N}$ such that for all $\ell > N$ & $m > N$

$$\left.\begin{array}{l} \sup\limits_{x \in (a,b)} \left| f_1^{(\ell)}(x) - f_1^{(m)}(x) \right| < \varepsilon \\ \sup\limits_{x \in (a,b)} \left| f_2^{(\ell)}(x) - f_2^{(m)}(x) \right| < \varepsilon \\ \vdots \\ \sup\limits_{x \in (a,b)} \left| f_n^{(\ell)}(x) - f_n^{(m)}(x) \right| < \varepsilon \end{array}\right\} \qquad (1.2.1)$$

the $i^{\underline{th}}$ relation of (1.2.1) shows that the sequence of the $i^{\underline{th}}$ components $\left\{ f_i^{(m)}(x) \right\}$ forms a fundamental sequence in $C(a,b)$, from the completeness of the space $C(a,b)$ [4] it follows that $f_i^{(m)}(x)$ converges as $m \longrightarrow \infty$ to an element $f_i(x) \in C(a,b)$.

Now consider the element $F(x) = \left\{ f_1(x), f_2(x), \ldots, f_n(x) \right\} \in C^n(a,b)$. We shall prove that it is the limit of the sequence $\left\{ F_m(x) \right\}$, in fact, let $\varepsilon > o$ be a given arbitrary number, then for the sequence $\left\{ f_i^{(m)}(x) \right\}$, $i = 1,2,\ldots,n$ there exists $N_i \in \mathbb{N}$ such that

for all $m > N_i$ $\| f_i^{(m)}(x) - f_i(x) \|_{C(a,b)} = \sup_{x \in (a,b)} | f_i^{(m)}(x) - f_i(x) | < \varepsilon$

Taking $N = \max (N_1, N_2, \ldots N_n)$, we then have for all $m > N$

$$\sup_{x \in (a,b)} | f_i^{(m)}(x) - f_i(x) | < \varepsilon \quad \text{for all } i \in \{1,2,\ldots n\}$$

this is equivalent to say that,

$$\max \left\{ \sup_{x \in (a,b)} | f_i^{(m)}(x) - f_i(x) | \right\} < \varepsilon, \ i = \{1,2,\ldots n\} .$$

i.e.

$$\| F_m(x) - F(x) \|_{C^n(a,b)} < \varepsilon$$

This proves that $\left\{ F_m(x) \right\}$ converges to $F(x)$ in $C^{(n)}(a,b)$

Compactness in $C^n(a,b)$

A set M of a metric space X is called relatively compact if, and only if, for all $\varepsilon > 0$ there exists a finite ε-net for M. If, moreover, M is closed, then it is called compact. Here we give the necessary and sufficient conditions for compactness of a set M of $C^{(n)}(a,b)$.

Theorem 1.2.2.

A set $M \subset C^n(a,b)$ is compact if, and only if,

1- M is uniformly bounded; i.e. $\exists$ a constant $K \in \mathbb{R}$ such that

$$\forall f \in M , \ \| f \|_{C^n(a,b)} < K$$